Section 1

Introduction

The classical calculus of variations has been generalized. The *maximum principle* for optimal control, developed in the late 1950s by L. S. Pontryagin and his co-workers, applies to all calculus of variations problems. In such problems, optimal control gives equivalent results, as one would expect. The two approaches differ, however, and the optimal control sometimes affords insights into a problem that might be less readily apparent through the calculus of variations.

Optimal control also applies to problems for which the calculus of variations is not convenient, such as those involving constraints on the derivatives of the functions sought. For instance, one can solve problems in which net investment or production rates are required to be nonnegative. While proof of the maximum principle under full generality is well beyond our scope, the now-familiar methods are used to generate some of the results of interest and to lend plausibility to others.

In optimal control problems, variables are divided into two classes, *state* variables and *control* variables. The movement of state variables is governed by first order differential equations. The simplest control problem is one of selecting a piecewise continuous control function $u(t)$, $t_0 \le t \le t_1$, to

$$\max \quad \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \tag{1}$$

$$\text{subject to} \quad x'(t) = g(t, x(t), u(t)), \tag{2}$$

$$t_0,\ t_1,\ x(t_0) = x_0 \text{ fixed}; \quad x(t_1) \text{ free}. \tag{3}$$

Here f and g are assumed to be known and continuously differentiable functions of three independent arguments, none of which is a derivative.

The *control* variable $u(t)$ must be a piecewise continuous function of time. The *state* variable $x(t)$ changes over time according to the differential equation (2) governing its movement. The control u influences the objective (1), both directly (through it own value) and indirectly through its impact on the evolution of the state variable x (that enters the objective (1)). The highest derivative appearing in the problem formulation is the first derivative, and it appears *only* as the left side of the *state equation* (2). (Equation (2) is sometimes also called the *transition* equation.) A problem involving higher derivatives can be transformed into one in which the highest derivative is the first, as will be shown later.

The prototypical calculus of variations problem of choosing a continuously differentiable function $x(t)$, $t_0 \le t \le t_1$, to

$$\begin{aligned} &\max \quad \int_{t_0}^{t_1} f(t, x(t), x'(t))\, dt \\ &\text{subject to} \quad x(t_0) = x_0 \end{aligned} \tag{4}$$

is readily transformed into an equivalent problem in optimal control. Let $u(t) = x'(t)$. Then the equivalent optimal control problem is

$$\begin{aligned} &\max \quad \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ &\text{subject to} \quad x'(t) = u(t), \quad x(t_0) = x_0. \end{aligned} \tag{5}$$

The state variable is x, while u is the control. For instance, our production planning Example I1.1 appears as

$$\begin{aligned} &\min \quad \int_0^T [c_1 u^2(t) + c_2 x(t)]\, dt \\ &\text{subject to} \quad x'(t) = u(t), \quad x(0) = 0, \quad x(T) = B, \quad u(t) \ge 0, \end{aligned} \tag{6}$$

where the production rate $u(t)$ is the control and the current inventory on hand $x(t)$ is the state variable. In this case, the objective is minimization rather than maximization, and the terminal point is fixed rather than free. These are typical variants from the initial format of (1) or (5).

While a calculus of variations problem (4) can always be put into an optimal control format (5), it is not always the most natural or useful form. For instance, Example I1.2 can be readily expressed as an optimal control problem:

$$\begin{aligned} &\max \quad \int_0^T e^{-rt} U(C(t))\, dt \\ &\text{subject to} \quad K' = F(K(t)) - C(t) - bK(t), \\ &\qquad\qquad\quad K(0) = K_0, \quad K(T) \ge 0, \quad C(t) \ge 0. \end{aligned} \tag{7}$$

Here $K(t)$ is the sole state variable; its rate of change is given in the differential equation. There is one control variable, the rate of consumption $C(t)$. Choice of $C(t)$ determines the rate of capital accumulation and also the value of the objective function.

Likewise, Example I1.3 is readily expressed as a problem of optimal control:

$$\begin{aligned} \max \quad & \int_0^T e^{-rt}[P(K(t)) - C(I(t))]\,dt \\ \text{subject to} \quad & K'(t) = I(t) - bK(t), \\ & K(0) = K_0, \quad K(T) \geq 0, \quad I(t) \geq 0. \end{aligned} \tag{8}$$

The objective is maximization of the discounted stream of profits, namely, revenues attainable with capital stock K less the cost of capital investment. Capital is augmented by gross investment but decays at exponential rate b. The state variable is K; the control variable is I.

An optimal control problem may have several state variables and several control variables. Each state variable evolves according to a differential equation. The number of control variables may be greater or smaller than the number of state variables.

The optimal control results are developed in the next sections for problems already solved by calculus of variations. This will develop familiarity with the new notations and tools. New problems will then be solved, and their use illustrated.

FURTHER READING

References on the techniques of optimal control theory include Pontryagin et al. (1962), Berkovitz (1974, 1976), Bryson and Ho, Fleming and Rishel, Hestenes, and Lee and Markus. In addition, there are a number of books that provide an introduction to the theory as well as discussion of applications in economics and management science; these include books by Hadley and Kemp, Intriligator (1971), and Takayama. For further surveys of applications in management science, see Bensoussan, Hurst, and Naslund; Bensoussan, Kleindorfer and Tapiero; and Sethi (1978). The references at the back of this book provide an overview of other applications to economics and management science; the list is merely suggestive of the range of work that has appeared.

Section 2

Simplest Problem—Necessary Conditions

The simplest problem in calculus of variations had both endpoints fixed. But the simplest problem in optimal control involves a free terminal point. To find necessary conditions that a maximizing solution $u^*(t), x^*(t)$, $t_0 \le t \le t_1$, to problem (1.1)–(1.3) must obey, we follow a procedure reminiscent of solving a nonlinear programming problem with Lagrange multipliers (see Section A5). Since the constraining relation (1.2) must hold at each t over the entire interval $t_0 \le t \le t_1$, we have a multiplier function $\lambda(t)$, rather than a single Lagrange multiplier value as would be associated with a single constraint. For now, let $\lambda(t)$ be any continuously differentiable function of t on $t_0 \le t \le t_1$; shortly, a convenient specification for its behavior will be made.

For any functions x, u satisfying (1.2) and (1.3) and any continuously differentiable function λ, all defined on $t_0 \le t \le t_1$, we have

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\,dt = \int_{t_0}^{t_1} [f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t)) - \lambda(t) x'(t)]\,dt, \quad (1)$$

since the coefficient of $\lambda(t)$ must be zero if (1.2) is satisfied, as we assume. Integrate by parts the last term on the right of (1)

$$-\int_{t_0}^{t_1} \lambda(t) x'(t)\,dt = -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} x(t)\lambda'(t)\,dt. \quad (2)$$

Substituting from (2) into (1) gives

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\,dt = \int_{t_0}^{t_1} [f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t)) + x(t)\lambda'(t)]\,dt - \lambda(t_1)x(t_1) + \lambda(t_0)x(t_0). \quad (3)$$

A control function $u(t)$, $t_0 \le t \le t_1$, together with the initial condition (1.3) and the differential equation (1.2) determine the path of the corresponding state variable $x^*(t)$, $t_0 \le t \le t_1$. Thus we may speak of finding the control function, since a corresponding state function is implied. Since the choice of control function $u(t)$ determines the state variable $x(t)$, choice of $u(t)$ thereby determines the value of (3) as well.

To develop the necessary conditions for solution of the calculus of variations problem (1.4), we constructed a one-parameter family of comparison curves $x(t) + ah(t), x'(t) + ah'(t)$, where $h(t)$ was arbitrary but fixed. In the current notation (1.5), $x' = u$ and a modified control function $u(t) + ah'(t)$ produces, via integration, a modified state function $x(t) + ah(t)$. However, for the implicit state equation (1.2), one cannot give an explicit expression for the modified control. Hence the modified state function will be expressed implicitly. Since the previous h, h' notation is not helpful here, we depart from the previous usage of h and now let $h(t)$ represent a fixed modification in the *control* $u(t)$.

We consider a one-parameter family of comparison controls, $u^*(t) + ah(t)$, where $u^*(t)$ is the optimal control, $h(t)$ is some fixed function, and a is a parameter. Let $y(t,a)$, $t_0 \le t \le t_1$, denote the state variable generated by (1.2) and (1.3) with control $u^*(t) + ah(t)$, $t_0 \le t \le t_1$. We assume that $y(t,a)$ is a smooth function of both its arguments. The second argument enters parametrically. Clearly $a = 0$ provides the optimal path x^*. Further, all comparison paths satisfy the initial condition. Hence

$$y(t,0) = x^*(t), \qquad y(t_0, a) = x_0. \tag{4}$$

With the functions u^*, x^*, and h all held fixed, the value of (1.1) evaluated along the control function $u^*(t) + ah(t)$ and corresponding state $y(t,a)$ depends on the single parameter a. Thus we write

$$J(a) = \int_{t_0}^{t_1} f(t, y(t,a), u^*(t) + ah(t))\, dt.$$

Using (3),

$$J(a) = \int_{t_0}^{t_1} [\, f(t, y(t,a), u^*(t) + ah(t)) + \lambda(t) g(t, y(t,a), u^*(t) + ah(t))$$
$$+ y(t,a)\lambda'(t)]\, dt - \lambda(t_1) y(t_1, a) + \lambda(t_0) y(t_0, a). \tag{5}$$

Since u^* is a maximizing control, the function $J(a)$ assumes its maximum at $a = 0$. Hence $J'(0) = 0$. Differentiating with respect to a and evaluating at $a = 0$ gives, on collecting terms,

$$J'(0) = \int_{t_0}^{t_1} [(f_x + \lambda g_x + \lambda') y_a + (f_u + \lambda g_u) h]\, dt - \lambda(t_1) y_a(t_1, 0) = 0, \tag{6}$$

where f_x, g_x and f_u, g_u denote the partial derivatives of the functions f, g with respect to their second and third arguments respectively; and y_a is the partial derivative of y with respect to its second argument. Since $a = 0$, the functions are evaluated along $(t, x^*(t), u^*(t))$. The last term of (5) is independent of a—that is $y_a(t_0, a) = 0$—since $y(t_0, a) = x_0$ for all a.

To this point, the function $\lambda(t)$ was required only to be differentiable. Since the precise impact of modifying the control variable on the course of the state variable (i.e., y_a) is difficult to determine, $\lambda(t)$ is selected to eliminate the need to do so. Let λ obey the linear differential equation

$$\begin{aligned}\lambda'(t) &= -\left[f_x(t, x^*, u^*) + \lambda(t) g_x(t, x^*, u^*) \right],\\ \lambda(t_1) &= 0.\end{aligned} \tag{7}$$

(Recall that x^* and u^* are fixed functions of t.) With λ given in (7), (6) holds provided that

$$\int_{t_0}^{t_1} \left[f_u(t, x^*, u^*) + \lambda g_u(t, x^*, u^*) \right] h \, dt = 0 \tag{8}$$

for any arbitrary function $h(t)$. In particular, it holds for $h(t) = f_u(t, x^*, u^*) + \lambda(t) g_u(t, x^*, u^*)$, so that

$$\int_{t_0}^{t_1} \left[f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t)) \right]^2 dt = 0. \tag{9}$$

This, in turn, implies the necessary condition that

$$f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t)) = 0, \qquad t_0 \le t \le t_1. \tag{10}$$

To sum up, we have shown that if the functions $u^*(t), x^*(t)$ maximize (1.1), subject to (1.2) and (1.3), then there is a continuously differentiable function $\lambda(t)$ such that u^*, x^*, λ simultaneously satisfy the *state equation*

$$x'(t) = g(t, x(t), u(t)), \qquad x(t_0) = x_0, \tag{11}$$

the *multiplier equation*

$$\lambda'(t) = -\left[f_x(t, x(t), u(t)) + \lambda(t) g_x(t, x(t), u(t)) \right], \qquad \lambda(t_1) = 0, \tag{12}$$

and the *optimality condition*

$$f_u(t, x(t), u(t)) + \lambda(t) g_u(t, x(t), u(t)) = 0. \tag{13}$$

for $t_0 \le t \le t_1$. The multiplier equation (12) is also known as the *costate*, *auxiliary*, *adjoint*, or *influence* equation.

The device for remembering, or generating these conditions (similar to solving a nonlinear programming problem by forming the Lagrangian, differentiating, etc.) is the *Hamiltonian*

$$H(t, x(t), u(t), \lambda(t)) \equiv f(t, x, u) + \lambda g(t, x, u). \tag{14}$$

Now

$\partial H/\partial u = 0$ generates (13): $\qquad \partial H/\partial u = f_u + \lambda g_u = 0;$ (13′)

$-\partial H/\partial x = \lambda'$ generates (12): $\qquad \lambda'(t) = -\partial H/\partial x = -(f_x + \lambda g_x);$ (12′)

$\partial H/\partial \lambda = x'$ recovers (11): $\qquad x' = \partial H/\partial \lambda = g.$ (11′)

In addition, we have $x(t_0) = x_0$ and $\lambda(t_1) = 0$. At each t, u is a stationary point of the Hamiltonian for the given values of x and λ. One can find u as a function of x and λ from (13) and substitute into (12) and (11) to get a system of two differential equations in x and λ. These conditions (11)–(13) are also necessary for a minimization problem.

For a maximization problem, it is also necessary that $u^*(t)$ maximize $H(t, x^*(t), u, \lambda(t))$ with respect to u. Thus, $H_{uu}(t, x^*, u^*, \lambda) \le 0$ is necessary for maximization. In a minimization problem, $u^*(t)$ must minimize $H(t, x^*(t), u, \lambda(t))$ with respect to u, and therefore $H_{uu}(t, x^*, u^*, \lambda) \ge 0$ is necessary. These two results have not yet been proved but they will be discussed later.

Example 1. Show that the necessary conditions for optimality in (1.5) are equivalent to the Euler equation

$$f_x = df_{x'}/dt \tag{15}$$

and transversality condition

$$f_{x'} = 0 \qquad \text{at } t_1 \tag{16}$$

that must be obeyed for the equivalent calculus of variations problem (1.4). What are the second order necessary conditions?

Form the Hamiltonian, following (14),

$$H = f(t, x, u) + \lambda u.$$

Then,

$$\partial H/\partial u = f_u + \lambda = 0, \tag{17}$$

$$\lambda' = -\partial H/\partial x = -f_x, \qquad \lambda(t_1) = 0. \tag{18}$$

Thus, if x^*, u^* are optimal, there must be a continuously differentiable function λ such that x^*, u^*, λ simultaneously satisfy $x' = u$, $x(t_0) = x_0$ and (17)–(18) over the interval $t_0 \le t \le t_1$.

To show that these conditions are equivalent to (15)–(16), differentiate (17) with respect to time:

$$df_u/dt + \lambda' = 0$$

and use the result to eliminate λ' from (18):

$$f_x = df_u/dt,$$

which is (15). Also, the boundary condition $\lambda(t_1) = 0$ is the same as in (16). Finally, the necessary condition $H_{uu} = f_{uu}(t, x^*, u^*) \le 0$ corresponds to the necessary Legendre condition $f_{x'x'}(t, x^*, x^{*\prime}) \le 0$.

Thus we have no new result. Optimal control yields, as necessary conditions, a system of two first order differential equations instead of the Euler single second order differential equation. The transversality conditions and second order necessary conditions under each formulation are likewise equivalent. In each case, the boundary conditions for solution of the differential equations are split, with one holding at the initial moment and the other holding at the final moment.

To show that (15) and (16) imply (17) and (18), we need only reverse the process. That is, define

$$\mu(t) = -f_{x'}(t, x(t), x'(t)). \tag{19}$$

Differentiate (19) with respect to t and substitute into (15):

$$f_x = -\mu'(t). \tag{20}$$

Putting (19) into (16) gives

$$\mu(t_1) = 0. \tag{21}$$

But (19)–(21) correspond exactly to conditions (17) and (18). Thus the two approaches yield equivalent necessary conditions for optimality, as claimed.

Since optimal control is equivalent to calculus of variations for all problems to which the latter applies, one may wonder why it is useful to learn about optimal control. One response is that it applies to a wider class of problems, to be studied later. Another answer is that optimal control may be more convenient for certain problems and may also suggest economic interpretations that are less readily apparent in solving by the calculus of variations. Each of these points will be illustrated; see Examples 2 and 3 below.

Example 2

$$\max \quad \int_0^1 (x+u)\,dt \tag{22}$$

$$\text{subject to} \qquad x' = 1 - u^2, \quad x(0) = 1. \tag{23}$$

Form the Hamiltonian

$$H(t, x, u, \lambda) = x + u + \lambda(1 - u^2).$$

Necessary conditions are (23) and

$$H_u = 1 - 2\lambda u = 0, \qquad H_{uu} = -2\lambda \le 0, \tag{24}$$

$$\lambda' = -H_x = -1, \qquad \lambda(1) = 0. \tag{25}$$

Solve (25) to find

$$\lambda = 1 - t. \tag{26}$$

Then $H_{uu} = -2(1-t) \le 0$ for $0 \le t \le 1$. Also, from (24),

$$u = 1/2\lambda = 1/2(1-t). \tag{27}$$

Substituting (27) into (23) gives

$$x' = 1 - 1/4(1-t)^2, \qquad x(0) = 1.$$

Integrating, using the boundary condition, and drawing the results together yields the solution:

$$x(t) = t - 1/4(1-t) + 5/4,$$
$$\lambda(t) = 1 - t,$$
$$u(t) = 1/2(1-t).$$

Example 3. The rate at which a new product can be sold at any time t is $f(p(t))g(Q(t))$ where p is the price and Q is the *cumulative* sales. We assume $f'(p) < 0$; sales varies inversely with price. Also $g'(Q) \gtrless 0$ for $Q \lessgtr Q_1$. For a given price, current sales grow with past sales in the early stage as people learn about the good from past purchasers. But as cumulative sales increase, there is a decline in the number of people who have not yet purchased the good. Eventually the sales rate for any given price falls, as the market becomes saturated. The unit production cost c may be constant or may decline with cumulative sales if the firm learns how to produce less expensively with experience: $c = c(Q)$, $c'(Q) \le 0$. Characterize the price policy $p(t)$, $0 \le t \le T$, that maximizes profits from this new "fad" over a fixed horizon T.

The problem is

$$\max \quad \int_0^T [p - c(Q)] f(p) g(Q)\, dt \tag{28}$$

$$\text{subject to} \qquad Q' = f(p)g(Q), \quad Q(0) = Q_0 > 0. \tag{29}$$

Price p is the control variable and cumulative sales Q is the state variable.

Form the Hamiltonian

$$H = f(p)g(Q)[p - c(Q) + \lambda]. \tag{30}$$

The optimal solution must satisfy (29) and

$$H_p = g(Q)\{f'(p)[p - c(Q) + \lambda] + f(p)\} = 0, \tag{31}$$

$$H_{pp} = g(Q)\{f''(p)[p - c(Q) + \lambda] + 2f'(p)\} \le 0, \tag{32}$$

$$\lambda' = -H_Q = f(p)\{g(Q)c'(Q) - g'(Q)[p - c(Q) + \lambda]\}, \tag{33}$$

$$\lambda(T) = 0. \tag{34}$$

We use these conditions to characterize the solution qualitatively. Since $g > 0$, we know from (31) that

$$\lambda = -f/f' - p + c. \tag{35}$$

Differentiating (35) totally with respect to t gives

$$\lambda' = -p'\left[2 - ff''/(f')^2\right] + c'Q'. \tag{36}$$

Substituting (35) into (32) and (33) gives

$$gf'\left[2 - f''f/(f')^2\right] \le 0, \tag{32'}$$

$$\lambda' = f[gc' + g'f/f']. \tag{33'}$$

Equate (36) and (33′), using (29):

$$\left[2 - ff''/(f')^2\right]p' = -g'f^2/f', \tag{37}$$

from which we conclude that

$$\operatorname{sign} p' = \operatorname{sign} g' \tag{38}$$

since $f' < 0$ and (32′) holds. Result (38) tells us that in marketing a good whose demand is as assumed here, the optimal price rises while the market is expanding ($Q < Q_1$) and falls as the market matures ($Q > Q_1$).

EXERCISES

1. Use optimal control to find the shortest distance between the point $x(a) = A$ and the line $t = b$.

2. Solve by optimal control

   $$\min \int_0^T [x^2(t) + ax(t) + bu(t) + cu^2(t)]\,dt$$

   subject to $x'(t) = u(t)$, $x(0) = x_0$ fixed, T fixed, $x(T)$ free, $c > 0$.

3. $$\max \int_1^5 (ux - u^2 - x^2)\,dt$$

   subject to $x' = x + u$, $x(1) = 2$.

4. Find necessary conditions for solution of

   $$\max \int_{t_0}^{t_1} f(t, x, u)\,dt$$

   subject to $x' = g(t, x, u)$, t_0, t_1 fixed, $x(t_0)$, $x(t_1)$ free.

5. $$\min \int_0^1 u^2(t)\,dt$$

   subject to $x'(t) = x(t) + u(t)$, $x(0) = 1$.

6. Show that necessary conditions for solution of

   $$\max \int_{t_0}^{t_1} f(t, x, u)\,dt + \varphi(x_1)$$

   subject to $x'(t) = g(t, x, u)$, $x(t_0) = x_0$, t_0, t_1 fixed, $x(t_1) = x_1$ free,

   are (11)–(13) except that $\lambda(t_1) = \varphi'(x_1)$. Relate to the corresponding transversality condition in the calculus of variations for the case $g = u$.

7. Using the results of exercise 6,

$$\min \int_0^1 u^2(t)\,dt + x^2(1)$$

$$\text{subject to} \qquad x'(t) = x(t) + u(t), \quad x(0) = 1.$$

8. The problem

$$\max \int_{t_0}^{t_1} f(x,u)\,dt$$

$$\text{subject to} \qquad x' = g(x,u), \quad x(t_0) = x_0 \text{ fixed}, \quad x(t_1) = x_1 \text{ free},$$

is *autonomous* since there is no explicit dependence on t. Show that the Hamiltonian is a constant function of time along the optimal path. [Hint: Compute using the chain rule,

$$dH/dt = H_x x' + H_u u' + H_\lambda \lambda'$$

and substitute from the necessary conditions for the partial derivatives of H.] Autonomous problems and their advantages are discussed further in Section 8.

9. The Euler equation for the calculus of variations problem

$$\max \int_{t_0}^{t_1} F(x,x')\,dt$$

$$\text{subject to} \quad x(t_0) = x_0 \text{ fixed}, \quad x(t_1) = x_1 \text{ free}.$$

can be written

$$F - x'F_{x'} = \text{const}, \qquad t_0 \le t \le t_1.$$

Show that this is equivalent to the condition that the Hamiltonian for the related control problem is constant.

10. Use the results of Exercise 8 to show that for Example 3,
    a. if $f(p) = e^{-ap}$, then the optimal sales rate fg is constant.
    b. if $f(p) = p^{-a}$, then revenue (pfg) is constant in an optimal program.

11. Discuss how the calculus of variations could be used to analyze Examples 2 and 3.

FURTHER READING

See (B4.7)–(B4.9) on the equivalence between a single second order differential equation and a pair of first order differential equations.

Example 3 was stimulated by Robinson and Lakhani, who also provided numerical results for a special case with discounting.

Compare the results mentioned here with those above, especially of Sections I8 and I11. Note that while the simplest problem of calculus of variations had a fixed endpoint, the simplest problem of optimal control has a free end value.

Section 3

Sufficiency

When are the necessary conditions for optimality both necessary and sufficient? In nonlinear programming, the Kuhn–Tucker necessary conditions are also sufficient provided that a concave (convex) objective function is to be maximized (minimized) over a closed convex region. In the calculus of variations, the necessary conditions are also sufficient for optimality if the integrand $F(t, x, x')$ is concave (convex) in x, x'. Analogous results obtain for optimal control problems.

Suppose that $f(t, x, u)$ and $g(t, x, u)$ are both differentiable concave functions of x, u in the problem of

$$\max \quad \int_{t_0}^{t_1} f(t, x, u)\, dt \tag{1}$$

$$\text{subject to} \qquad x' = g(t, x, u), \quad x(t_0) = x_0. \tag{2}$$

The argument t of $x(t)$ and $u(t)$ will frequently be suppressed. Suppose that the functions x^*, u^*, λ satisfy the necessary conditions

$$f_u(t, x, u) + \lambda g_u(t, x, u) = 0, \tag{3}$$

$$\lambda' = -f_x(t, x, u) - \lambda g_x(t, x, u), \tag{4}$$

$$\lambda(t_1) = 0, \tag{5}$$

and the constraints of (2) for all $t_0 \le t \le t_1$. Suppose further that x and λ are continuous with

$$\lambda(t) \ge 0 \tag{6}$$

for all t in case $g(t, x, u)$ is nonlinear in x or in u, or both. Then the functions x^*, u^* solve the problem given by (1) and (2). Thus if the functions f and g are both jointly concave in x, u (and if the sign restriction in (6) holds), then the necessary conditions (2)–(5) are also sufficient for optimality.

The assertion can be verified as follows. Suppose that x^*, u^*, λ satisfy (2)–(6). Let x, u be functions satisfying (2). Let f^*, g^*, and so on denote functions evaluated along (t, x^*, u^*) and let f, g, and so on denote functions evaluated along the feasible path (t, x, u). Then we must show that

$$D \equiv \int_{t_0}^{t_1} (f^* - f)\, dt \geq 0. \tag{7}$$

Since f is a concave function of (x, u), we have

$$f^* - f \geq (x^* - x)f_x^* + (u^* - u)f_u^*, \tag{8}$$

and therefore (reasons to follow)

$$\begin{aligned} D &\geq \int_{t_0}^{t_1} [(x^* - x)f_x^* + (u^* - u)f_u^*]\, dt \\ &= \int_{t_0}^{t_1} [(x^* - x)(-\lambda g_x^* - \lambda') + (u^* - u)(-\lambda g_u^*)]\, dt \\ &= \int_{t_0}^{t_1} \lambda [g^* - g - (x^* - x)g_x^* - (u^* - u)g_u^*]\, dt \\ &\geq 0, \end{aligned} \tag{9}$$

as was to be shown. The second line of (9) was obtained by substituting from (4) for f_x^* and from (3) for f_u^*. The third line of (9) was found by integrating by parts the terms involving λ', recalling (2) and (5). The last line follows from (6) and the assumed concavity of g in x and u.

If the function g is linear in x, u, then λ may assume any sign. The demonstration follows since the last square bracket in (9) will equal zero. Further, if f is concave while g is convex and $\lambda \leq 0$, then the necessary conditions will also be sufficient for optimality. The proof proceeds as shown above, except that in the next to last line λ and its coefficient are each nonpositive, therefore making their product nonnegative.

EXERCISES

1. Show that if f and g are both concave functions of x and u, if (6) holds, and if x^*, u^*, λ satisfy (2)–(5), then $u^*(t)$ does maximize the Hamiltonian $H(t, x^*(t), u, \lambda(t))$ at each t, $t_0 \leq t \leq t_1$, with respect to u.

2. Show that if minimization was required in problem (1) and (2), and if the functions f and g are both jointly convex in x, u, then functions x^*, u^*, λ satisfying (2)–(6) will solve the problem. Also show that $u^*(t)$ will minimize the Hamiltonian $H(t, x^*(t), u, \lambda(t))$ at each t.

3. Investigate whether the solutions obtained in the exercises of Section 2 minimize or maximize.

4. Suppose $\varphi(x_1)$ is a concave function and that $f(t,x,u)$ and $g(t,x,u)$ are differentiable concave functions of (x,u). State and prove a sufficiency theorem for

$$\max \quad \int_{t_0}^{t_1} f(t,x,u)\,dt + \varphi(x_1)$$

$$\text{subject to} \qquad x' = g(t,x,u), \quad x(t_0) = x_0,\ t_0, t_1 \text{ fixed}, \quad x(t_1) = x_1 \text{ free}.$$

FURTHER READING

Mangasarian has provided the basic sufficiency theorem for optimal control. See also Section 15 and Seierstad and Sydsaeter for extensions to more complex control problems.

Compare the present results with those of Section I6.

Section 4

Interpretations

The multiplier λ in optimal control problems has an interesting and economically meaningful interpretation. In nonlinear programming the Lagrange multiplier is interpreted as a marginal valuation of the associated constraint. (See Section A5.) Here $\lambda(t)$ is the marginal valuation of the associated state variable at t.

Consider

$$\max \int_{t_0}^{t_1} f(t, x, u)\, dt$$
$$\text{subject to} \quad x'(t) = g(t, x, u), \qquad x(t_0) = x_0. \tag{1}$$

Let $V(x_0, t_0)$ denote the maximum of (1), for a given initial state x_0 at initial time t_0. Let x^*, u^* be the state and control functions providing this maximum, and let λ be the corresponding multiplier. Suppose u^* is a continuous function of t.

We also consider a modification of problem (1) in which the initial state is $x_0 + a$, where a is a number close to zero. The maximum for the modified problem is $V(x_0 + a, t_0)$. Let x and u denote the state and control functions providing this maximum.

Appending the differential equation in (1) with a continuously differentiable multiplier function $\lambda(t)$ gives

$$V(x_0, t_0) = \int_{t_0}^{t_1} f(t, x^*, u^*)\, dt$$
$$= \int_{t_0}^{t_1} [\, f(t, x^*, u^*) + \lambda g(t, x^*, u^*) - \lambda x'\,]\, dt. \tag{2}$$

Integrate the last term by parts (recalling (2.2)):

$$V(x_0, t_0) = \int_{t_0}^{t_1} (f^* + \lambda g^* + \lambda' x)\,dt - \lambda(t_1)x^*(t_1) + \lambda(t_0)x(t_0), \quad (3)$$

where asterisks label functions evaluated along (t, x^*, u^*). Similarly, one finds that (using the same λ)

$$V(x_0 + a, t_0) = \int_{t_0}^{t_1} f\,dt$$

$$= \int_{t_0}^{t_1} (f + \lambda g + \lambda' x)\,dt - \lambda(t_1)x(t_1) + \lambda(t_0)[x(t_0) + a],$$

where x, u are optimal for this problem. Subtracting,

$$V(x_0 + a, t_0) - V(x_0, t_0) = \int_{t_0}^{t_1} [f(t, x, u) - f(t, x^*, u^*)]\,dt$$

$$= \int_{t_0}^{t_1} (f + \lambda g + \lambda' x - f^* - \lambda g^* - \lambda' x^*)\,dt$$

$$+ \lambda(t_0)a - \lambda(t_1)[x(t_1) - x^*(t_1)]. \quad (4)$$

Expand the integrand by Taylor's theorem around (t, x^*, u^*):

$$V(x_0 + a, t_0) - V(x_0, t_0) = \int_{t_0}^{t_1} [(f_x^* + \lambda g_x^* + \lambda')(x - x^*)$$

$$+ (f_u^* + \lambda g_u^*)(u - u^*)]\,dt$$

$$+ \lambda(t_0)a - \lambda(t_1)[x(t_1) - x^*(t_1)] + \text{h.o.t.}$$

$$(5)$$

Let λ be the multiplier satisfying the necessary conditions for (1). Since x^*, u^*, λ satisfy the necessary conditions (2.11)–(2.13) for optimality,

$$\lambda' = -(f_x^* + \lambda g_x^*), \qquad f_u^* + \lambda g_u^* = 0, \qquad \lambda(t_1) = 0,$$

(5) reduces to

$$V(x_0 + a, t_0) - V(x_0, t_0) = \lambda(t_0)a + \text{h.o.t.} \quad (6)$$

Divide (6) by the parameter a and then let a approach zero:

$$\lim_{a \to 0} [V(x_0 + a, t_0) - V(x_0, t_0)]/a = V_x(x_0, t_0) = \lambda(t_0) \quad (7)$$

provided the limit exists. The first equation of (7) constitutes the definition of derivative of V with respect to x. We assume that this derivative exists. Thus the multiplier $\lambda(t_0)$ is the marginal valuation in the optimal program of the state variable at t_0.

The discussion thus far has only considered the initial time. However, $\lambda(t)$ is the marginal valuation of the associated state variable at time t. If there were an exogenous, tiny increment to the state variable at time t and if the problem were modified optimally thereafter, the increment in the total value of the objective would be at the rate $\lambda(t)$.

To verify this assertion, recall that the objective in (1) is additive. Any portion of an optimal program is itself optimal.

For instance, suppose we follow the solution x^*, u^* to (1) for some period $t_0 \le t \le t^{\#}$ and then stop and reconsider the optimal path from that time on forward:

$$\begin{aligned} &\max \quad \int_{t^{\#}}^{t_1} f(t,x,u)\,dt \\ &\text{subject to} \qquad x'(t) = g(t,x,u), \quad x(t^{\#}) = x^*(t^{\#}). \end{aligned} \tag{8}$$

A solution to (8) must be $x^*(t), u^*(t)$, $t^{\#} \le t \le t_1$, namely, the same as the original solution to (1) on the interval from $t^{\#}$ forward. To see this, suppose it untrue. Then there is a solution to (8) providing a larger value than does x^*, u^* on $t^{\#} \le t \le t_1$. The value of (1) could then be improved by following x^*, u^* to $t^{\#}$ and switching to the solution to (8). But this contradicts the assumed optimality of x^*, u^* for (1). Therefore, x^*, u^*, $t^{\#} \le t \le t_1$ must solve (8).

We return to the question of the interpretation of λ. Application of the methods used to reach (7) to problem (8) leads to the result

$$V_x(x(t^{\#}), t^{\#}) = \lambda(t^{\#}), \tag{9}$$

provided that this derivative exists, where λ is the function associated with problem (1) (since the solutions to (1) and (8) coincide on $t^{\#} \le t \le t_1$). Thus $\lambda(t^{\#})$ is the marginal valuation of the state variable at $t^{\#}$. But the time $t^{\#}$ was arbitrary, so for any t, $t_0 \le t \le t_1$,

$$V_x(x(t), t) = \lambda(t), \qquad t_0 \le t \le t_1, \tag{10}$$

is the marginal valuation of the state variable at time t, whenever this derivative exists.

It is easy to confirm the interpretation at t_1. If there is no salvage term, the marginal value of the state at terminal time is zero: $\lambda(t_1) = 0$. And if there is a salvage term, the marginal value of the state is the marginal contribution of the state to the salvage term: $\lambda(t_1) = \varphi'(x_1)$ (recall Exercise 2.16).

For ease in discussion, let x be the stock of an asset and $f(t,x,u)$ the current profit. It is an identity that

$$\begin{aligned} \lambda(t_1)x(t_1) &= \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} (x'\lambda + x\lambda')\,dt \\ &= \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} [d(x\lambda)/dt]\,dt. \end{aligned} \tag{11}$$

Recall that $\lambda(t)$ is the marginal valuation of the state variable at t. Thus the value of the terminal stock of assets is the value of the original stock plus the change in the value of assets over the control period $[t_0, t_1]$. The total rate of change in the value of assets

$$d(x\lambda)/dt = x'\lambda + x\lambda'$$

is composed of the value of additions (or reductions) in the stock of assets plus the change in the value of existing assets. That is, both changes in amount of assets held and in the unit value of assets contribute to the change in the value of all assets. From (3), the rate at which the total value accumulates is

$$f + \lambda g + x\lambda' = H + x\lambda' \qquad \text{where} \quad H = f + \lambda g. \tag{12}$$

The first term is the direct gain at t of $f(t, x, u)$, say the current cash flow. The second term is an indirect gain through the change in the state variable. One can think of $\lambda g = \lambda x'$ as the increase in future profitability attributable to the increase in the stock of assets. The third and remaining term $x\lambda'$ represents the changed valuation in current assets, the capital gains. Thus, (12) represents the contribution rate at t, both direct and indirect, toward the total value.

At each moment, one chooses the control u to maximize the net contribution (12) toward total value. For given state variable $x(t)$ and marginal valuation of the state $\lambda(t)$, this means choosing $u(t)$ to maximize H, and hence to satisfy

$$\partial H/\partial u = f_u + \lambda g_u = 0, \qquad t_0 \le t \le t_1, \tag{13}$$

and also

$$\partial^2 H/\partial u^2 = f_{uu} + \lambda g_{uu} \le 0. \tag{14}$$

Note also that if one were free to choose x to maximize (12), then one would set

$$f_x + \lambda g_x + \lambda' = 0. \tag{15}$$

Of course, the choice of u completely determines x. A sufficient condition for x^*, u^*, λ to be optimal is that they be feasible with $\lambda(t_1) = 0$ and that the problem

$$\max_{x,u} \left[H(x, u, \lambda(t), t) + \lambda'(t)x \right] \tag{16}$$

have $x = x^*(t)$, $u = u^*(t)$ as its solution for all $t_0 \le t \le t_1$.

Example. Let $P(x)$ be the profit rate that can be earned with a stock of productive capital x, where $P'(0) > 0$ and $P'' < 0$. The capital stock decays at a constant proportionate rate $b \ge 0$. Investment cost is an increasing convex function of the gross investment rate u, with $C'(0) = 0$ and $C'' > 0$. We seek the investment rate $u(t)$ that maximizes the present value of the profit stream over the fixed planning period $0 \le t \le T$:

$$\max \int_0^T e^{-rt}[P(x) - C(u)]\,dt \tag{17}$$

$$\text{subject to} \qquad x' = u - bx, \quad x(0) = x_0 > 0, \quad u \ge 0. \tag{18}$$

We assume that the nonnegativity condition on u will be met automatically. Therefore, since the capital stock is initially positive, the capital cannot become negative; review (18). (This is Problem 1.8.)

The Hamiltonian is

$$H = e^{-rt}[P(x) - C(u)] + \lambda(u - bx).$$

Optimal functions x, u, and λ satisfy (18) and

$$H_u = -e^{-rt}C'(u) + \lambda = 0, \tag{19}$$

$$H_{uu} = -e^{-rt}C''(u) < 0, \tag{20}$$

$$\lambda' = -H_x = -e^{-rt}P'(x) + b\lambda, \qquad \lambda(T) = 0. \tag{21}$$

Condition (20) is satisfied by our assumptions on C. Equation (19) states that, at each t, the marginal cost of investment must equal the marginal value λ of a unit of capital. Both terms are discounted back to the moment $t = 0$ of planning. Equivalently,

$$C'(u(t)) = e^{rt}\lambda(t) \tag{22}$$

gives condition that the marginal cost equal the marginal benefit, in values at t.

Differential equation (21) can be manipulated to show the composition of the marginal value of a unit of capital. Subtract $b\lambda$ from each side, multiply by integrating factor e^{-bt}, and integrate using the boundary condition in (21):

$$e^{-bt}\lambda(t) = \int_t^T e^{-(r+b)s}P'(x(s))\,ds.$$

Therefore, the value at time t of a marginal unit of capital is the discounted stream of marginal profits it generates:

$$e^{rt}\lambda(t) = \int_t^T e^{-(r+b)(s-t)}P'(x(s))\,ds. \tag{23}$$

The calculation reflects the fact that capital decays, and therefore at each time $s > t$ it contributes only a fraction e^{-bs} of what a whole unit of capital would add. Combining (22) and (23) yields the marginal cost–marginal benefit condition for optimal investment

$$C'(u(t)) = \int_t^T e^{-(r+b)(s-t)}P'(x(s))\,ds. \tag{24}$$

EXERCISE

Derive condition (24) using the calculus of variations.

FURTHER READING

These interpretations are greatly elaborated by Dorfman. The sufficiency condition related to (16) is due to Seierstad and Sydsaeter; see also Section 15. Benveniste and Scheinkman (1979) give sufficient conditions for the value function to be differentiable.