CS 67800, Spring 2017/18

Problem Set 2: Markov Networks

Submission Date : Thursday 17/5/18, 23:59

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1. Prove that for a Bayesian Network $\mathcal{G}$: $\mathcal{I}(\mathcal{M}[\mathcal{G}]) \subseteq \mathcal{I}(\mathcal{G})$, where $\mathcal{M}[\mathcal{G}]$ is the moral graph of $\mathcal{G}$. In other words: $\mathcal{M}[\mathcal{G}]$ is an I-map for $\mathcal{G}$.
   Bonus: Prove that $\mathcal{M}[\mathcal{G}]$ is a minimal I-map.

   Answer:
   Assume by contradiction that there exists $\texttt{Sep}_{\mathcal{M}[\mathcal{G}]}(X\,;Y \mid \mathbf{Z})$ but $(X \perp Y \mid \mathbf{Z}) \notin \mathcal{I}_{d-sep}(\mathcal{G})$. So there exists an active trail from X to Y given $\mathbf{Z}$ in $\mathcal{G}$. This trail also exists in $\mathcal{M}[\mathcal{G}]$ (by construction).

   If there are no v-structures along this trail, then none of its nodes are in $\mathbf{Z}$ (otherwise the trail would not be active in $\mathcal{G}$). So this trail is also active in $\mathcal{M}[\mathcal{G}]$, in contradiction.

   On the other hand, for any v-structure on that trail in $\mathcal{G}$, say $X_{i-1} - X_i - X_{i+1}$, with X_i or one of its descendants in $\mathbf{Z}$ (since the path is active in $\mathcal{G}$). Therefore, $\mathcal{M}[\mathcal{G}]$ has an edge $X_{i-1} - X_{i+1}$ (by construction), and the path is active in $\mathcal{M}[\mathcal{G}]$, in contradiction.

2. (a) Show an example for a distribution P (not necessarily positive) and a Markov Network $\mathcal{H}$, such that P satisfies $\mathcal{I}_{LM}(\mathcal{H})$, but not $\mathcal{I}(\mathcal{H})$ (i.e. not $\mathcal{I}_{sep}(\mathcal{H})$).
   (b) Show an example for a distribution P (not necessarily positive) for which the intersection property:
   $$(X \perp Y \mid Z, W) \wedge (X \perp W \mid Z, Y) \Longrightarrow (X \perp Y, W \mid Z)$$
   does not hold.

3. We say that a node X in $\mathcal{H}$ is *eliminable* if all its neighbors are connected (i.e. the node and its neighbors form a clique).
   (a) Show that a chordal graph has an eliminable node.
   (b) Show that if you eliminate that node (and the edges connected to it), the remaining graph is still chordal, and still has an eliminable node.

4. *(I-Maps and P-Maps)* Consider a probability distribution P over the variables A, B, C and D that satisfies only the following independencies:
   1. $A \perp C|B$.
   2. $A \perp C|B, D$.
   3. $A \perp D|B$.
   4. $A \perp D|B, C$.
   5. $B \perp D$.
   6. $B \perp D|A$.
   7. $A \perp D$.

   (a) Draw a Bayesian network that is a perfect map for P.
   Answers: The 2 graphs below are both perfect maps for P. Either of them is valid, and the other is the I-equivalent graph for part (b).

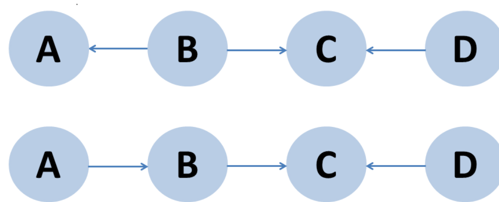

Figure 1: BN graphs for (a)

(b) Does this perfect map have any I-equivalent graphs? If so, draw them. If not, explain why not.

Answers: Yes. See the solution to part (a) for the graph.

(c) Draw a Markov network that is a minimal I-map for P and explain why the Markov network is or is not also a perfect map.

Answers: A Markov net with these undirected edges and additional $B - D$ edge

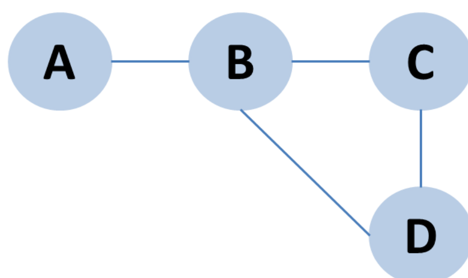

Figure 2: MN for (c)

is a valid I-map. This Markov net is not a perfect map because B and D are not independent in this network, while they are independent in the original probability distribution.