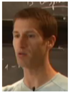
Finiteness for Hecke algebras of p-adic groups.

A theorem by Jean-Francois Dat, David Helm, Robert Kurinczuk, and Gilbert Moss

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Required Background:
basic representation theory of p-adic groups -

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Question (to the audience)

What happens in general?

Simple reductions

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- Reduction to the case $R = \mathbb{Z}_\ell\langle\sqrt{p}\rangle$ – fathfully flat desent
- Localizing the problem – A-finit objects
- Reduction to "parabolic induction of cuspidal representations preseres finitness" – Bernstein decomposition.

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Let $R \to S$ be a ring homomorphism.

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- *If $A \otimes_R S$ is S–finite type and S is faithfully flat over R then A is R–finite type*

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From now we will fix $R = \mathbb{Z}_\ell\langle\sqrt{p}\rangle$.

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Proof of the reduction.

Take

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3. *M is B-finite then so is N.*

Reduction to representations induced from cuspidal

Proposition

any $V \in \mathcal{M}_R^{f.g.}(G)$ can be embedded

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So, it is enough to show

Theorem ([DHKM])

Parabolic induction of a cuspidal $\mathfrak{Z}$-finite representation is $\mathfrak{Z}$-finite.

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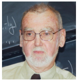

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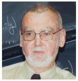

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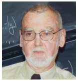

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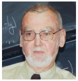

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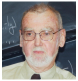

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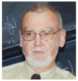

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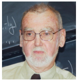

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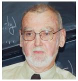

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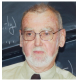

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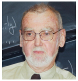

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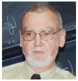

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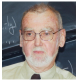

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   $\forall$ Levi $M < G$, and $V \in \mathcal{M}_R(M)$ the following diagram is commutative:

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Fargues-Scholze Theory as a black box (cont)

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for large enough n.

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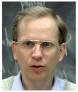

What is (roughly) $E(\mathcal{G})$

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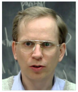

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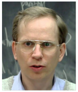

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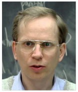

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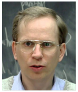

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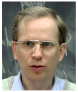

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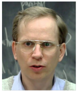

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Definition

Define $E_n(\mathcal{G}) := O(Hom(W_F^0/P_F^n, \mathcal{G})//Ad(\mathcal{G}))$

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