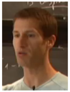
Finiteness for Hecke algebras of p-adic groups.

A theorem by Jean-Francois Dat, David Helm, RobertT Kurinczuk, and Gilbert Moss

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We reduce the main result to showing:
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- Enough to show induced from cuspidal $\mathfrak{Z}_R(M)$-finite, is $\mathfrak{Z}_R(G)$-finite

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- Formulated (UA) and (FC) to show Splitting of all cuspidals of a Levi.

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For $\rho \in Irr_c(G)$, D_ρ splits $\mathcal{M}^{f.g.}(G)$

We show that D splits $\mathcal{M}(G)$. Notice $D|_{G^0} = \{\rho|_{G^0}\}$. Now use :

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Let $\mathcal{M}$ be an abelian category and let $P \in Obj(\mathcal{M})$ be a compact projective generator. Let $R = End_{\mathcal{M}}(P)$. Then $\beta(X) = Hom_{\mathcal{M}}(P, X)$ is an equivalence from $\beta : \mathcal{M} \to Mod^r(R)$ (right R-modules).

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Corollary (Bernstein)

$\mathcal{M}(G)(D)$ is Noetherian (a submodule of f.g is f.g).

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3. Construction of Splitting

Splitting $\mathrm{Irr}_c(G)$ - Uniform admissibility

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Given r commuting $N \times N$ matrices $A_1, A_2, ..., A_r \in M_N(\mathbb{C})$ we have $\dim_{\mathbb{C}}(< A_1, ..., A_r >) \leq C(r) N^{2-\epsilon}, \epsilon = \frac{1}{2^{r-1}}$.

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Proof of Uniform Admissibility.

By Burnside: $\rho : \mathcal{H}(G, K) \to End(V^K)$ is onto. Let $\mathcal{H}(G, K) = \mathcal{H}(K_0, K)C\mathcal{H}(K_0, K)$ with $d = dim(\mathcal{H}(K_0, K))$ and

$$C := Span_{\mathbb{C}}\{a(\lambda) = e_{K\lambda K} : \lambda \in \Lambda^+\}$$

Key calculation:

$$N^2 = \dim(V^K)^2 = \dim(\rho(\mathcal{H}(G, K)) \leq d^2 \dim(\rho(C)) \leq d^2 C(r)N^{2-\epsilon}$$

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Theorem (Finiteness of cuspidals)

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- *For any $\lambda \in \Lambda^+$ the operator $a(\lambda) : V^K \to V^K$ is nilpotent.*
- *There exists a set $S(G, K, V) \subset G$ that is compact modulo the center $Z(G)$ with the following property:*

$$Supp(D_{K,v}) \subset S(G, K, V)$$

for all $v \in V^K$. Here $D_{K,v}(g) = \pi(e_K)\pi(g^{-1})v$.

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We use $a(\lambda) = e_{K\lambda K} = \delta_\lambda e_{\lambda^{-1}K\lambda} e_K$ to get

$$V^K \cap \cup_{n=1}^{\infty} Ker(a(\lambda)^n) = V^K \cap V(U_\lambda) = V^K$$

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Using UA we obtain

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Fix G, F and $K < G$. Then the number of orbits D in $Irr_c(G)$ with $D^K \neq \varnothing$ is finite.

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Theorem (i preserve f.g.)

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Exercise (using Geometric Lemma).

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Theorem (Bernstein)

The decomposition $\mathcal{M}(G) = \Pi_\Omega \mathcal{M}(\Omega)$ induces $\mathfrak{Z}(G) = \Pi\mathfrak{Z}(\Omega)$. We have $\mathfrak{Z}(\Omega) \simeq \mathcal{O}(\Omega)$ given by action.

Let $\Omega = [(M, D)]$ then we have $\Pi(D) = c - ind_{G^0}^{G}(D|_{G^0})$ and $\Pi(\Omega) = i_{G,M}(\Pi(D)$ a f.g. module.

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Theorem (Stabilization Lemma)

For $K < G$ there exists $C(G, K)$ such that for any $P = MU, \lambda \in \Lambda^+(M, K)$ any $V \in \mathcal{M}(G)$ and any $n \geq C(G, K)$ the map

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For V cuspidal (not necessarily irreducible) we can deduce it from uniform admissibility. The general case requires another lecture.