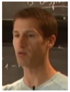
Finiteness for Hecke algebras of p-adic groups – finiteness on the Galois side.

A theorem by Jean-Francois Dat, David Helm, Robert Kurinczuk, and Gilbert Moss

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Lift f to Γ. Let G be as above.
Then $Hom(\Gamma, G)//G^\circ \xrightarrow{\phi \mapsto \phi(f)} G//G^\circ$ *is finite.*

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Lemma (Cancellation)

Assume:

$$\begin{tikzcd} X \arrow[r, "\text{dominant}"] \arrow[rrr, bend right, "\text{finite}"'] & Y \arrow[r] & Z \arrow[r] & W \end{tikzcd}$$

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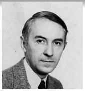
non-connected Chevalley (iso)morphism

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Fix a pinning $(T_G(\bar{k}), \ldots)$ *of* $G^\circ(\bar{k})$.

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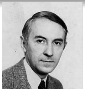  

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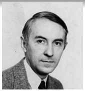  

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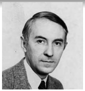  

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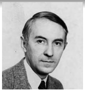

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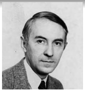  

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Proof.

By induction, it is enough to show that $(G_x)^\circ \not\subset Z(G)$. $\square$

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Corollary

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Question to the audience: What happens for general finite P with $(|P|, \ell) = 1$?

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