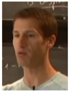
Finiteness for Hecke algebras of p-adic groups.

A theorem by Jean-Francois Dat, David Helm, Robert Kurinczuk, and Gilbert Moss

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- We can assume $R = \mathbb{Z}_\ell\langle\sqrt{p}\rangle$.

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3. *V is D-finite then so is W.*

The main Theorem is equivalent to the following one:

Theorem ([DHKM])

Any f.g. $V \in \mathcal{M}_R(G)$ is $\mathfrak{Z}(G) := \mathfrak{Z}_R(G) := \mathfrak{Z}(\mathcal{M}_R(G))$-finite.

Reduction to representations induced from cuspidal

Theorem (Bernstein*)

any $V \in \mathcal{M}_R^{f.g.}(G)$ *can be embedded*

$$V \subset \bigoplus i_{M_i}^G(W_i),$$

for some $M_i \overset{Levi}{<} G$ *and* $\mathfrak{Z}(Z(M_i))$*-finite (cusp.)* $W_i \in \mathcal{M}(M_i)$.

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Theorem ([DHKM])

Let $M < G$ *be a Levi. Let* $W \in \mathcal{M}_R(M)$ *be* $\mathfrak{z}(Z(M))$*-finite representation.*

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Theorem ([DHKM])

Let $M < G$ be a Levi. Let $W \in \mathcal{M}_R(M)$ be $\mathfrak{Z}(Z(M))$-finite representation. Then $i_M^G(W) \in \mathcal{M}_R(G)$ is $\mathfrak{Z}(G)$-finite.

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Proposition

In the situation above, $i_M^G(W)$ is $\mathfrak{z}(Z(M))$-finite.

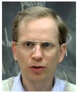
Proof.

$$i_M^G(W)^K = \bigoplus_{[x] \in K \backslash G/P} W^{xKx^{-1} \cap P}$$

$\square$

The excursion algebra

Let $\mathcal{G}$ be an R-group scheme



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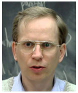
Let $\mathcal{G}$ be an R-group scheme with a (finite) action of Gal_F.



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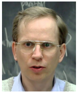

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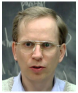

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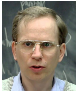

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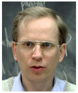

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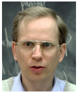

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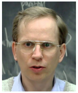

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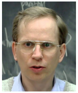

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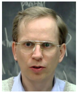

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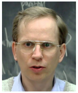

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- $E_n(\mathcal{G})_{red} \hookrightarrow O(Z^1(\Gamma_n, \mathcal{G}))^{\mathcal{G}} = O(Z^1(\Gamma_n, \mathcal{G})//\mathcal{G})$

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Proof.

$$\begin{tikzcd} E_n(\mathcal{G})_{red} \arrow[r] & E_n(\mathcal{H})_{red} \\ O(\mathcal{G}//\widetilde{Ad}(\mathcal{G})) \arrow[u] \arrow[r, "finite"] & O(\mathcal{H}//\widetilde{Ad}(\mathcal{H})) \arrow[u, "finite"'] \end{tikzcd}$$

□

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E(\widehat{M}^{ab}) \arrow[u] \arrow[r, "\sim", "FS_{Z(M)^\circ}"'] & \mathfrak{z}(Z(M)^\circ) \arrow[u] &
\end{tikzcd}
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For large enough n.

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Corollary

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Corollary

$$\begin{tikzcd} E(\widehat{G}) \arrow[r] \arrow[rr, bend left, "FS_R"] & E(\widehat{G})_{red} \arrow[r] & \mathfrak{z}_R(G) \end{tikzcd}$$

Proof of the main theorem

$$
\begin{tikzcd}
E(\widehat{G}) \arrow[r, "FS_G"] \arrow[dd] & \mathfrak{z}(G) \arrow[r] & End(i_M^G(W)) \arrow[r, bend left] & i_M^G(W) \\
 & E_n(\widehat{M}) \arrow[dr] & & \\
E(\widehat{M}) \arrow[ur, two heads] \arrow[r, "FS_M"] & \mathfrak{z}(M) \arrow[r] & End(W) \arrow[uu] & \\
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\end{tikzcd}
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& E_n(\widehat{M}) \arrow[dr] & & \\
E(\widehat{M}) \arrow[ur, two heads] \arrow[r, "FS_M"] & \mathfrak{Z}(M) \arrow[r] & End(W) \arrow[uu] & \\
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\begin{tikzcd}
E(\widehat{G}) \arrow[r, "FS_G"] \arrow[ddd] \arrow[dr, two heads] & {\color{red}\mathfrak{z}(G)} \arrow[r] & End(i_M^G(W)) \arrow[r, bend left] & i_M^G(W) \\
 & E_n(\widehat{G}) \arrow[ur] \arrow[d] & & \\
 & E_n(\widehat{M}) \arrow[dr] & & \\
E(\widehat{M}) \arrow[r, "FS_M"] \arrow[ur, two heads] & \mathfrak{z}(M) \arrow[r] & End(W) \arrow[uuu] & \\
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 & E_n(\widehat{G})_{red} \arrow[d] \arrow[ur] & & \\
 & E_n(\widehat{M})_{red} \arrow[dr] & & \\
 & & & \\
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& E_n(\widehat{M})_{red} \arrow[ddr] & & \\
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Excursion Algebra E_n vs. $O(Hom(\Gamma_n, G)//G)$

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