Exercise 2

1. The transpose of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order, as rows. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Now, let:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \bar{v} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}, \bar{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

For all the following products, decide whether the product is well-defined, and if it is – calculate the result:

1.1. $A\bar{v}$
1.2. $D^T\bar{u}$
1.3. AD
1.4. DA
1.5. A^TA
1.6. D^TA

(make sure you understand how the calculation you do relates to the formulas we learned in class for the general term $[A\vec{v}]_i$ and the general term $[AB]_{ij}$)

3.7 What can you conclude from (3.3) and (3.4) about multiplying a matrix with a diagonal matrix from the left and from the right?

2. Let A be a 3×3 matrix. Find a matrix B such that the i^{th} column of AB is equal to the i^{th} column of A, only multiplied by i. That is, the 1st column of AB is the same as that of A, the 2nd column of AB is the same as the 2nd column of A, only multiplied by 2, and so on.

3. Use the definition of the transpose of a matrix from question 1 and prove that $(AB)^T = B^T A^T$.
Guidance: Denote by a_{ij} and b_{ij} the general entries of A, B. Show that the general entry of $(AB)^T$ is equal to the general entry of $B^T A^T$.

4. Thinking of matrices.
4.1. Find two non-zero matrices A, B such that $AB = 0$.
4.2. Find two matrices A, B such that $AB = 0$ but $BA \neq 0$.
4.3. Find a matrix A such that $AA = A$, but A is not the identity matrix I.

5. Prove the following statement or give a counter example:

$$tr(AB) = tr(A)tr(B)$$

6. Matrix powers and the inverse matrix.
Let A, P be $n \times n$ matrices, and assume P is regular. Show that for all $m \in \mathbb{N}$:

$$(P^{-1}AP)^m = P^{-1}A^m P$$

7. Definition: A square matrix A is called "nilpotent" if there exists an integer m such that $A^m = 0$ ("nilpotent" means it has the potential to become null, zero). The smallest m for which $A^m = 0$ is called the "nilpotency index" of A.

7.1. What is the nilpotency index of the matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$?

7.2. Can a nilpotent matrix by regular?

8. Show that $A = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i & \frac{2}{3}i \\ -\frac{2}{3}i & -\frac{1}{3} - \frac{2}{3}i \end{pmatrix}$ is unitary.

9. True or false?
9.1. All the diagonal entries of an Hermitian matrix are real.

10. Let $A = \begin{pmatrix} 3 - 5i & 2 + 4i \\ 6 + 7i & 1 + 8i \end{pmatrix}$. Find A^H.

11. Show that the following matrix is normal:

$$A = \begin{pmatrix} 2+3i & 1 \\ i & 1+2i \end{pmatrix}$$