Exercise for chapter 2 - solution

1.

$$v = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix},\ u = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix},\ A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

1.1. $A\bar{v} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ -26 \\ 17 \end{bmatrix}$

1.2. $D^T\bar{u} =$ not defined

1.3. $AD = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -6 & 15 \\ 3 & -12 & -10 \\ 18 & -3 & 0 \end{bmatrix}$

1.4. $DA = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 9 \\ -3 & -12 & 6 \\ 30 & 5 & 0 \end{bmatrix}$

1.5. $A^TA = \begin{bmatrix} -1 & 1 & 6 \\ 2 & 4 & 1 \\ 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 & -5 \\ 8 & 21 & -2 \\ -5 & -2 & 13 \end{bmatrix}$

1.6. $D^TA = DA = \begin{bmatrix} -3 & 6 & 9 \\ -3 & -12 & 6 \\ 30 & 5 & 0 \end{bmatrix}$

1.7. From the left- rescales the rows. From the right- rescales the columns.

2. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3.

Question (Ex02, Q6)

Use the definition of the transpose of a matrix from question 3 and prove that $(AB)^T = B^T A^T$.
Guidance: Denote by a_{ij} and b_{ij} the general entries of A, B. Show that the general entry of $(AB)^T$ is equal to the general entry of $B^T A^T$.

Solution

The general term for the ij element in a composite matrix is:

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Therefore, for $(AB)^T$ we will have to switch between the columns and rows and get:

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

Now, for the $B^T A^T$ part, we know that $[B^T]_{ij} = b_{ji}$ and $[A^T]_{ij} = a_{ji}$. So using the general term for the ij element in a composite matrix, we get:

$$[B^T A^T]_{ij} = \sum_{k=1}^{n} [B^T]_{ik} [A^T]_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

Clearly, we found that the two general terms are equal, as we wanted.

Solution to question 3 (Schaum, page 44):

2.13. Prove Theorem 2.3(iv): $(AB)^T = B^T A^T$.

Let $A = [a_{ik}]$ and $B = [b_{kj}]$. Then the ij-entry of AB is

$$a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$

This is the ji-entry (reverse order) of $(AB)^T$. Now column j of B becomes row j of B^T, and row i of A becomes column i of A^T. Thus, the ij-entry of $B^T A^T$ is

$$[b_{1j}, b_{2j}, \ldots, b_{mj}][a_{i1}, a_{i2}, \ldots, a_{im}]^T = b_{1j} a_{i1} + b_{2j} a_{i2} + \cdots + b_{mj} a_{im}$$

Thus, $(AB)^T = B^T A^T$ on because the corresponding entries are equal.

4.

4.1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

4.2. $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. $AB = 0, BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

4.3. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

5. We can use the example from 4.1 as a counter example: $tr(AB) = 0 \neq tr(A) * tr(B) = 1$

6.

Solution

$$(P^{-1}AP)^m = \underbrace{(P^{-1}AP)(P^{-1}AP)\dots(P^{-1}AP)}_{m\ times} = P^{-1}A\underbrace{PP^{-1}}_{I}A\underbrace{PP^{-1}}_{I}\dots\underbrace{PP^{-1}}_{I}AP = P^{-1}A^mP$$

7.

7.1. m=3.

7.2. No. Because for a nilpotent matrix there is m such that $\det(A^m) = 0$. but, $\det(A^m) = \det(A)^m$. So $\det(\mathrm{A}) = 0$, which means that A is singular.

8. $A = \begin{bmatrix} \frac{1}{3} - \frac{2}{3}i & \frac{2}{3}i \\ -\frac{2}{3}i & -\frac{1}{3} - \frac{2}{3}i \end{bmatrix}$ → $\overline{A^T} = \begin{bmatrix} \frac{1}{3} + \frac{2}{3}i & \frac{2}{3}i \\ -\frac{2}{3}i & -\frac{1}{3} + \frac{2}{3}i \end{bmatrix}$. $\overline{A^T}A = I$.

9. True. For Hermitian matrix $a_{ij} = \overline{a_{ji}}$. On the diagonal: $a_{ii} = \overline{a_{ii}}$ so a_{ii} are real.

10. $A = \begin{bmatrix} 3-5i & 2+4i \\ 6+7i & 1+8i \end{bmatrix}$ −> $A^H = \begin{bmatrix} 3+5i & 6-7i \\ 2-4i & 1-8i \end{bmatrix}$

11. $A = \begin{bmatrix} 2+3i & 1 \\ i & 1+2i \end{bmatrix}$ → $\overline{A^T} = \begin{bmatrix} 2-3i & -i \\ 1 & 1-2i \end{bmatrix}$. $\overline{A^T}A = A\overline{A^T} = \begin{bmatrix} 14 & 4-4i \\ 4+4i & 6 \end{bmatrix}$.