Exercise for chapter 3 - solution

1. Check in Matlab/Wolfarm Alpha

2. Check in Matlab/Wolfarm Alpha

3. $\bar{v}$ cannot be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$

1. 3.1. An equivalent question to the question whether $\bar{v}$ can be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$ is: is there a solution to this matrix equation:
$$\begin{bmatrix} | & | & | \\ \overline{u_1} & \overline{u_2} & \overline{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{v}$$
2. Det=0 so the matrix is not invertible and the system of equations has no solution at all, meaning that indeed, $\bar{v}$ cannot be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$

4. Ex02, 2018

$$x + 2y + az = -3 - a$$
$$x + (2 - a)y - z = 1 - a$$
$$ax + ay + z = 6$$

R2->R2-R1

$$x + 2y + az = -3 - a$$
$$-ay - (a + 1)z = 4$$
$$ax + ay + z = 6$$

R3->R3+R2

$$x + 2y + az = -3 - a$$
$$-ay - (a + 1)z = 4$$
$$ax - az = 10$$

If a=0 then we have no solution.

Otherwise:
Subtitute $az = ax - 10$ in eq 1:

$$x + 2y + ax - 10 = -3 - a$$
$$-ay - (a + 1)z = 4$$
$$ax - az = 10$$

$$x(a + 1) + 2y = 7 - a$$
$$-ay - (a + 1)z = 4$$
$$ax - az = 10$$

If a=-1:

$$2y = 8 \to y = 4$$
$$-x + z = 10$$

So there's a solution for a=-1 and any z: (z-10,4,z).

For any other value ($a \neq 0, -1$):

$$x(a + 1) + 2y = 7 - a$$
$$-ay - (a + 1)z = 4$$
$$ax - az = 10$$

To eliminate y, R2->R2*2, R1->R1*a:

$$xa(a + 1) + 2ay = a(7 - a)$$
$$-2ay - 2(a + 1)z = 8$$
$$ax - az = 10$$

Now **Add** Eq1 and Eq2, and move az in Eq3:

$$xa(a + 1) - 2(a + 1)z = a(7 - a) + 8$$
$$ax = 10 + az$$

Plug in ax in from R3 to R1:

$$(10 + az)(a + 1) - 2(a + 1)z = a(7 - a) + 8$$

So:

$$z(a^2+a)+10(a+1)-2(a+1)z=a(7-a)+8$$
$$z(a^2-a-2)=a(7-a)+8-10(a+1)$$
$$z(a^2-a-2)=7a-a^2+8-10a-10$$
$$z(a^2-a-2)=-a^2-2-3a$$
$$z(a-2)(a+1)=-(a^2+3a+2)$$
$$z(a-2)(a+1)=-(a^2+3a+2)$$
$$z(a-2)(a+1)=-(a+1)(a+2)$$
We can divide by $(a+1)$ because we know $a\neq -1$:
$$z(2-a)=a+2$$
If $a\neq 2$, then $z=\frac{a+2}{2-a}$, and we can go look for x,y.

But **if $\boldsymbol{a=2}$**, let's go back to the original 3 equations and check what happens:
$$x(a+1)+2y=7-a$$
$$-ay-(a+1)z=4$$
$$ax-az=10$$
Which means:
$$3x+2y=5$$
$$-2y-3z=4$$
$$x-z=5$$
So plug in $z=x-5$ from R3 to R2:
$$3x+2y=5$$
$$-2y-3(x-5)=4$$
So:
$$3x+2y=5$$
$$-2y-3x+15=4$$
Or:
$$3x+2y=5$$
$$-3x-2y=-11$$
Which means we **there is no solution for $\boldsymbol{a=2}$.**

5. 3.1 and 3.2. Schaum, page 92 (question 3.12).

Question 5.1:

(a) Apply "Replace L_2 by $-3L_1 + L_2$" and "Replace L_3 by $-2L_1 + L_3$" to eliminate x from the second and third equations. This yields

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \\ 3x_3 + 6x_4 - 9x_5 &= 3 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \end{aligned}$$

(We delete L_3, because it is a multiple of L_2.) The system is in echelon form with pivot variables x_1 and x_3 and free variables x_2, x_4, x_5.

To find the parametric form of the general solution, set $x_2 = a$, $x_4 = b$, $x_5 = c$, where a, b, c are parameters. Back-substitution yields $x_3 = 1 - 2b + 3c$ and $x_1 = 3a + 5b - 8c$. The general solution is

$$x_1 = 3a + 5b - 8c, \;\; x_2 = a, \;\; x_3 = 1 - 2b + 3c, \;\; x_4 = b, \;\; x_5 = c$$

or, equivalently, $u = (3a + 5b - 8c, a, 1 - 2b + 3c, b, c)$.

Question 5.2:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ 2x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ 5x_1 + 12x_2 - 7x_3 + 6x_4 &= 3 \end{aligned}$$

So to make R2 easier (R2->R2-2R1):

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ x_2 + 4x_3 - 7x_4 &= -3 \\ 5x_1 + 12x_2 - 7x_3 + 6x_4 &= 3 \end{aligned}$$

Now make R3 easier (R3->R3-5R1):

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ x_2 + 4x_3 - 7x_4 &= -3 \\ 2x_2 + 8x_3 - 14x_4 &= -7 \end{aligned}$$

Now R3->R3-2R2:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ x_2 + 4x_3 - 7x_4 &= -3 \\ 0 &= -1 \end{aligned}$$

Which means there is no solution.

Another way to see that this is going to a "no solution" case, is to note that the matrix rows are dependent:

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -2 & 1 \\ 5 & 12 & -7 & 6 \end{pmatrix}$$

2R2 + R1 = R3, so the rank of the matrix is 2.

Because the row rank is equal to the column rank, we can deduce that the column space is 2D, which calls for the possibility of being unable to find a solution.

For those who struggles with 3.2. Let's take a look at the following (inconsistent system):

$$x + y + z = 10$$
$$x + y + 2z = 20$$
$$3x + 3y + 5z = 30$$

The Wrong Way

Take R2-R1 to get:

$$x + y + z = 10$$
$$z = 10$$
$$3x + 3y + 5z = 30$$

Let's take R3-5*R2:

$$x + y + z = 10$$
$$z = 10$$
$$-2x - 2y = -20 \quad \rightarrow \quad x + y = 10$$

Set $y = a$ as a free variable and get the general solution:

$$x = 10 - a$$
$$(10 - a, a, 10)$$

We used all the equations, but actually out "solution" is wrong. Let's set $a = 0$ and see what we get in the first equation:

$$10 + 10 = 10$$

What did we do wrong? Well, by substituting $y = a$ and "finding" x, we actually said that we know how x depends on all other variables. But that's not true: because we still had an equation (Eq. 1) that said something else about the dependence of x on all other variables. This is why Gaussian elimination works – it actually finds the "one true dependence" of each variable on the other variables.

The Right Way

$$x + y + z = 10$$
$$x + y + 2z = 20$$
$$3x + 3y + 5z = 30$$

Get rid of x in R2 (take R2-R1) and R3 (take R3-3R1):

$$x + y + z = 10$$
$$z = 10$$
$$2z = 0$$

Extra Question

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \end{aligned}$$

Now, Let's find a particular solution. Set all the free variables to 0:

$$x_1 + 2x_3 = 2$$
$$x_3 = 1$$

Substituting x_3 in the first equation we get $x_1 = 0$.
Hence (0,0,1,0,0) is a particular solution to the system.
Now look for a general solution of the **homogeneous** system, continuing from the reduced form we found, and setting each free variable as a parameter ($x_2 = a, x_4 = b, x_5 = c$):

$$x_1 - 3a + 2x_3 - b + 2c = 0$$
$$x_3 + 2b - 3c = 0$$

We get:

$$x_3 = -2b + 3c$$

$$x_1 = -3a - 2x_3 + b + 2c = -3a - 2(-2b + 3c) + b + 2c = -3a + 5b - 4c$$

So the general solution to the homogeneous system is:

$$(-3a + 5b - 4c, a, -2b + 3c, b, c)$$

So the general solution to the nonhomogeneous system is:

$$(0,0,1,0,0) + (-3a + 5b - 4c, a, -2b + 3c, b, c)$$

6. false. Counter example- 5.2

7. Schaum, page 150 (question 3.11)

(c) Eliminate x from the second and third equations by the operations "Replace L_2 by $-2L_1 + L_2$" and "Replace L_3 by $-3L_1 + L_3$." This yields the new system $x +$

$$\begin{aligned} x + 2y - 3z &= 1 \\ y - 2z &= 2 \\ 2y - 4z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y - 3z &= 1 \\ y - 2z &= 2 \end{aligned}$$

(The third equation is deleted, because it is a multiple of the second equation.) The system is in echelon form with pivot variables x and y and free variable z.

To find the parametric form of the general solution, set $z = a$ and solve for x and y by back-substitution. Substitute $z = a$ in the second equation to get $y = 2 + 2a$. Then substitute $z = a$ and $y = 2 + 2a$ in the first equation to get

$$x + 2(2 + 2a) - 3a = 1 \qquad \text{or} \qquad x + 4 + a = 1 \qquad \text{or} \qquad x = -3 - a$$

Thus, the general solution is

$$x = -3 - a, \quad y = 2 + 2a, \quad z = a \qquad \text{or} \qquad u = (-3 - a, \quad 2 + 2a, \quad a)$$

where a is a parameter.

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8. Schaum, page 150 (question 3.11)

(b) Eliminate x from the second and third equations by the operations "Replace L_2 by $3L_1 + L_2$" and "Replace L_3 by $-5L_1 + L_3$." This gives the equivalent system

$$\begin{aligned} x + 2y - 3z &= -1 \\ 7y - 11z &= -10 \\ -7y + 11z &= 7 \end{aligned}$$

The operation "Replace L_3 by $L_2 + L_3$" yields the following degenerate equation with a nonzero constant:

$$0x + 0y + 0z = -3$$

This equation and hence the system have no solution.

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9. Schaum, page 90

Consider the system

$$x + ay = 4$$
$$ax + 9y = b$$

(a) For which values of a does the system have a unique solution?

(b) Find those pairs of values (a, b) for which the system has more than one solution.

(a) Eliminate x from the equations by forming the new equation $L = -aL_1 + L_2$. This yields the equation

$$(9 - a^2)y = b - 4a \tag{1}$$

The system has a unique solution if and only if the coefficient of y in (1) is not zero—that is, if $9 - a^2 \neq 0$ or if $a \neq \pm 3$.

(b) The system has more than one solution if both sides of (1) are zero. The left-hand side is zero when $a = \pm 3$. When $a = 3$, the right-hand side is zero when $b - 12 = 0$ or $b = 12$. When $a = -3$, the right-hand side is zero when $b + 12 - 0$ or $b = -12$. Thus, $(3, 12)$ and $(-3, -12)$ are the pairs for which the system has more than one solution.