Exercise 4

1. Show that the vectors $\overline{w}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \overline{w}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \overline{w}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ form a spanning set of $\mathbb{R}^3$.

Guidance: Show that a general vector $\overline{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ can be written as a linear combination of $\overline{w}_1, \overline{w}_2\ \overline{w}_3$.

2. Let $\overline{v} = \begin{pmatrix}2\\7\\8\end{pmatrix}$ and $\overline{u}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \overline{u}_2 = \begin{pmatrix}1\\3\\5\end{pmatrix}, \overline{u}_3 = \begin{pmatrix}1\\5\\9\end{pmatrix}$

2.1. Show that $\overline{v}$ **cannot** be written as a linear combination of $\overline{u}_1, \overline{u}_2, \overline{u}_3$.

2.2. Do $\overline{u}_1, \overline{u}_2, \overline{u}_3$ span $\mathbb{R}^3$?

3. Without any calculation, determine whether the following set of vectors in $\mathbb{R}^3$ is linearly dependent:

$\overline{u}_1 = \begin{pmatrix}1\\2\\5\end{pmatrix}, \overline{u}_2 = \begin{pmatrix}1\\3\\1\end{pmatrix}, \overline{u}_3 = \begin{pmatrix}2\\5\\7\end{pmatrix}, \overline{u}_4 = \begin{pmatrix}3\\1\\4\end{pmatrix}$

4. The zero vector is a vector in which all entries are 0. It is usually denoted by 0 (and not $\overline{0}$). For example, the zero vector in 2D is $\begin{pmatrix}0\\0\end{pmatrix}$.

Given the definition that we saw in class for linear dependence, prove that any set of vectors that includes the zero vector is linearly dependent.

5. Let: $\overline{v} = \begin{pmatrix}1\\3\end{pmatrix}, \overline{u} = \begin{pmatrix}-1\\0\end{pmatrix}, \overline{w} = \begin{pmatrix}-2\\-6\end{pmatrix}$

5.1. Draw the above vectors on the same coordinate system.

5.2. Add a schematic illustration of $Sp(\overline{v})$

5.3. Calculate $\overline{v} + 3\overline{u}$ and show that your answer agrees with the geometrical interpretation of vector summation.

6. Check your answer to question 2 in the main exercise for the specific vector $\overline{v} = \begin{pmatrix}5\\-6\\2\end{pmatrix}$

7. True/false? Justify your answers.

7.1. Let A be a real $m \times n$ matrix. $rowsp(A)$ is a subspace of $\mathbb{R}^m$.

7.2. Let A be a real $m \times n$ matrix. $rowsp(A)$ is a subspace of $\mathbb{R}^n$.

7.3. The solution set for a nonhomogeneous system $A\overline{x} = \overline{b}$ is a subspace of $\mathbb{R}^n$.

7.4. The set of real 2×2 matrices with zero determinant ($\{A \in M_{2\times2} | \det(A) = 0\}$) is a subspace of the space of real 2×2 matrices.

8. Let V be the vector space of real 2×2 matrices. Let U be the subspace of real symmetric 2×2 matrices.

8.1. Find $\dim(U)$.

To prove your answer, find a basis for U. Make sure to prove that it is indeed a basis (remember the two properties that every basis needs to fulfill).

9. Let $P_2(x)$ be the vector space of polynomials of degree smaller or equal to 2. The following polynomials form a basis for this space:

$$p_1 = x + 1$$
$$p_2 = x - 1$$
$$p_3 = x^2 - 2x + 1$$

Let $\bar{v} = 2x^2 - 5x + 9$ be a vector in this space. Find the coordinate vector (i.e., the coefficients) of $\bar{v}$ with respect to this new basis.

10. In question 2, you have shown that $\bar{v} = \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix}$ cannot be written as a linear combination of $\bar{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \bar{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \bar{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$.

10.1. Formulate this question as a matrix equation.

10.2. What is the determinant of the matrix whose columns are $\bar{u}_1, \bar{u}_2, \bar{u}_3$? Check your answer by direct calculation.