Exercise for chapter 4 - solution

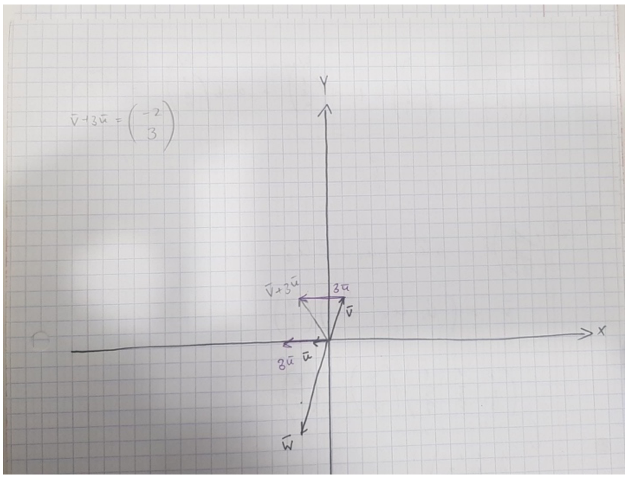
1. $\bar{v} = c\overline{w_1} + (b-c)\overline{w_2} + (a-b)\overline{w_3}$

2.
   a. Let's assume that $\bar{v}$ can be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$, meaning that there exist scalars x, y, z such that $\bar{v} = x\overline{u_1} + y\overline{u_2} + z\overline{u_3}$. trying to solve this system of eq. will end up with no solution (show it!).
   b. No. we found a vector in $\mathbb{R}^3$ that cannot be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$, so they cannot span $\mathbb{R}^3$.

3. They are because they are 4 vectors in a space of 3 dimensions.

4. One definition is that one or more of the vectors is inside the span of the remaining vectors. The zero vector is always inside of any span. Therefore, any set of vectors that includes the zero vector is linearly dependent.

5.



6. Same answer, $\bar{v}$ cannot be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$. This is because trying to solve the system of eq.: $\bar{v} = x\overline{u_1} + y\overline{u_2} + z\overline{u_3}$ will end up with no solution (show it!).

7. Final solutions (without explanation)
    a. False
    b. True
    c. False. (the zero vector doesn't belong to the solution set)
    d. False. Explanation:

Let $\mathcal{M}_n$ be the vector space of $n \times n$ matrices and let

$$S_n = \{A \in \mathcal{M}_n : \det A = 0\}$$

Then S_n is *not* a subspace of $\mathcal{M}_n$. Indeed, let

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Then $A, B \in S$ but $\det(A + B) = 1$ so $A + B \notin S_n$.

In particular, when $n = 2$, the subset S_2 is not a subspace of $\mathcal{M}_2$ because

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are elements of S_2 but $\det(A + B) = 1$ so $A + B \notin S_2$.

8. Let's call U: W.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be an arbitrary element in the subspace W.
Then since $A^{\mathrm{T}} = A$, we have

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

This implies that $a_{12} = a_{21}$, and hence

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \\ &= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Let $B = \{v_1, v_2, v_3\}$, where v_1, v_2, v_3 are 2×2 matrices appearing in the above linear combination of A.
Note that these matrices are symmetric.
Hence we showed that any element in W is a linear combination of matrices in B.
Thus B is a spanning set for the subspace W.

We show that B is linearly independent.
Suppose that we have

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $c_1 = c_2 = c_3 =$ and the set B is linearly independent.

As B is a linearly independent spanning set, we conclude that B is a basis for the subspace W.

Recall that the dimension of a subspace is the number of vectors in a basis of the subspace.

In part (b), we found that $B = \{v_1, v_2, v_3\}$ is a basis for the subspace W.
As B consists of three vectors, the dimension of W is 3.

9. A solution which is useful for polynomials, is to equate the given polynomial with a linear combination of the basis vectors (a*p1+b*p2+c*p3). Then, group together the same powers, and equate the coefficients from both sides to get a,b and c.

EXAMPLE 4.16 Consider the vector space $\mathbf{P}_2(t)$ of polynomials of degree ≤ 2. The polynomials

$$p_1 = t + 1, \qquad p_2 = t - 1, \qquad p_3 = (t-1)^2 = t^2 - 2t + 1$$

form a basis S of $\mathbf{P}_2(t)$. The coordinate vector $[v]$ of $v = 2t^2 - 5t + 9$ relative to S is obtained as follows. Set $v = xp_1 + yp_2 + zp_3$ using unknown scalars x, y, z, and simplify:

$$\begin{aligned} 2t^2 - 5t + 9 &= x(t+1) + y(t-1) + z(t^2 - 2t + 1) \\ &= xt + x + yt - y + zt^2 - 2zt + z \\ &= zt^2 + (x + y - 2z)t + (x - y + z) \end{aligned}$$

Then set the coefficients of the same powers of t equal to each other to obtain the system

$$z = 2, \qquad x + y - 2z = -5, \qquad x - y + z = 9$$

The solution of the system is $x = 3$, $y = -4$, $z = 2$. Thus,

$$v = 3p_1 - 4p_2 + 2p_3, \text{ and hence, } [v] = [3, -4, 2]$$

10.

a. This is equivalent to ask whether the matrix eq. $\bar{v} = \begin{bmatrix} | & | & | \\ \overline{u_1} & \overline{u_2} & \overline{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has a solution.

b. Det = 0. Meaning that the system of equations either has no solution at all, or it has infinite solutions. Elementary row operations will end in this matrix eq.: $\bar{v} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which has no solution. So $\bar{v}$ cannot be written as a linear combination of $\overline{u_1}, \overline{u_2}, \overline{u_3}$.