Exercise for chapter 5-6 - solution

1. You need to show that $T\left(a * \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = a * T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$. You can do that; this is very straightforward.

2. This is the projection on the xz plane.
   a. XZ plane
   b. 2
   c. Y axis
   d. 1
   e. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. …
   a. No linear

$F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (xy, x)$

Let $v = (1, 2)$ and $w = (3, 4)$; then $v + w = (4, 6)$. Also,

$$F(v) = (1(2), 1) = (2, 1) \quad \text{and} \quad F(w) = (3(4), 3) = (12, 3)$$

Hence,

$$F(v + w) = (4(6), 4) = (24, 6) \neq F(v) + F(w)$$

   b. Linear
   c. Not linear because $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq 0$
   d. Not linear

$F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|, \; y + z)$

Let $v = (1, 2, 3)$ and $k = -3$. Then $kv = (-3, -6, -9)$. We have

$$F(v) = (1, 5) \quad \text{and} \quad kF(v) = -3(1, 5) = (-3, -15).$$

Thus,

$$F(kv) = F(-3, -6, -9) = (3, -15) \neq kF(v)$$

4. …
   a. Rows are independent→ rank=3. The null space and row space are orthogonal complements so dim(ker(A))=5-3=2. The left null space and column space are orthogonal complements so dim(ker(A^t))=3-3=0.
   b. Rows are dependent→ rank=2. The null space and row space are orthogonal complements so dim(ker(A))=4-2=2. The left null space and column space are orthogonal complements so dim(ker(A^t))=3-2=1.

5. Rank=dim(colsp(A))=dim(rowsp(A))=5. The null space and row space are orthogonal complements so dim(ker(A))=9-5=4. The left null space and column space are orthogonal complements so dim(ker(A^t))=7-5=2.

6. ....

Problem 1: A is an $m \times n$ matrix of rank r. Suppose there are right-hand-sides $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution.

(a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) $A^{\mathrm{T}}\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$. Why must this be true?

Solution (15 points = 10+5)

(a) First of all, the rank r of a matrix is the number of column (row) pivots, it must be less than equal to m and n. If the matrix were of full row rank, i.e., $r = m$, it would imply that $A\vec{x} = \vec{b}$ always has a solution; we know that this is not the case, and hence $r \neq m$. To sum up, the inequalities among m, n, r are $r \leq n, r < m$.

(b) Since A^{T} is an $n \times m$ matrix, the null space $N(A^{\mathrm{T}})$ has dimension $m - r$, which is positive by (a). Hence, $A^{\mathrm{T}}\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$.