Exercise 7

1. Let $\bar{v} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \bar{w} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$.

1.1. Calculate the projection of $\bar{v}$ onto $\bar{w}$.
1.2. Calculate the projection of $\bar{w}$ onto $\bar{v}$.

2. Let $\bar{u}, \bar{w} \in \mathbb{R}^2$ be two orthogonal vectors.
For this question only, assume that you don't know that orthogonal vectors satisfy $\bar{u}^T\bar{w} = 0$.

Use the Pythagorean theorem to show that $\bar{u}^T\bar{w} = 0$.

3. Let $\bar{x}$ be an $n \times 1$ vector and A be a $n \times n$ matrix.
What is the result of $\bar{x}^T A\bar{x}$? If it is a vector or matrix, write down the general term of the result. If it is a scalar, write down the result. You should use the Σ notation.

4. Let A be a matrix. Show that any vector $\bar{x} \in \ker(A)$ is orthogonal to any vector $\bar{a} \in rowsp(A)$ (remember, the row space of A is the space spanned by the rows of the matrix A).

5. Cauchy-Shcwartz inequality
The Cauchy-Shcwartz inequality says that for any two vectors $\bar{v}, \bar{w}$:

$$\langle \bar{v}, \bar{w} \rangle^2 \le \langle \bar{v}, \bar{v} \rangle \langle \bar{w}, \bar{w} \rangle \qquad \text{or equivalently:} \qquad |\langle \bar{v}, \bar{w} \rangle| \le \|\bar{v}\|\|\bar{w}\|$$

Here one bar means "absolute value" and two bars mean "vector norm".
Verify the Cauchy-Schwartz inequality for $\bar{u} = (1,0,3,2)^T, \bar{v} = (4,1,0,1)^T$.

6. Consider the following three vectors:

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix}$$

6.1. Show that they are pairwise orthogonal.
6.2. Show that they any vector $\bar{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ can be expressed by a linear combination of $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$. In other words (see chapter 4): show that they form a basis of $\mathbb{R}^3$.
6.3. Let $\bar{u} = \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix}$. Write down $\bar{u}$ as a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$. Use the fact that the given basis is an orthogonal basis.

7. Let $V = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ be a set of nonzero pairwise-orthogonal vectors in $\mathbb{R}^n$.

7.1. Prove that V is linearly independent.
7.2. Assume that $k = n$. Do the vectors in V form a basis for $\mathbb{R}^n$?

8. The following vectors are a basis for a subspace $V \subset \mathbb{R}^4$:

$$\bar{v}_1 = (1,1,1,1)^T \quad \bar{v}_2 = (1,2,4,5)^T \quad \bar{v}_3 = (1,-3,-4,-2)^T$$

Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

9. Orthogonal matrices

9.1. Prove that the product of two orthogonal matrices is an orthogonal matrix.

<u>Inner product spaces</u>

11. Prove that the inner product over the complex numbers is linear in the second term.

12. Prove that for any inner product space, the norm satisfies:

$$\left||k\bar{v}|\right| = |k| \cdot \left||\bar{v}|\right|$$