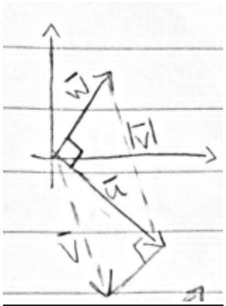
Exercise for chapter 7 – solution

1. ..

   a. $proj_{\bar{w}}\bar{v} = \frac{\bar{v}^T\bar{w}}{|\bar{w}|^2}\bar{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

   b. $proj_{\bar{v}}\bar{w} = \frac{\bar{w}^T\bar{v}}{|\bar{v}|^2}\bar{v} = \frac{-4}{11}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

2. Let $\bar{v} = \bar{u} - \bar{w}$.



Pythagorean theorem: $|\bar{u}|^2 + |\bar{w}|^2 = |\bar{v}|^2 = |\bar{u} - \bar{w}|^2$.
This gives us:
$\bar{u}^T\bar{u} + \bar{w}^T\bar{w} = (\bar{u} - \bar{w})^T(\bar{u} - \bar{w}) = (\bar{u}^T - \bar{w}^T)(\bar{u} - \bar{w})$=$\bar{u}^T\bar{u} - \bar{u}^T\bar{w} - \bar{w}^T\bar{u} + \bar{w}^T\bar{w}$
We can cancel similar terms and get: $-\bar{u}^T\bar{w} - \bar{w}^T\bar{u} = 0$
But we know that $\bar{u}^T\bar{w} = \bar{w}^T\bar{u}$ (this is a scalar)
So $2\bar{w}^T\bar{u} = 0$ and therefore $\bar{w}^T\bar{u} = 0$

3.
$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\bar{x}^T A \bar{x} =?$
$A\bar{x}$ is a nx1 vector and therefore $\bar{x}^T A\bar{x}$ is a scalar.

$$\bar{x}^T A \bar{x} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij}x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$$

4. $\bar{x} \in \ker(A)$ means that $A\bar{x} = \bar{0}$. From matrix multiplication we know that this means that every row of A multiplied by $\bar{x}$ equal 0. Meaning that $\bar{x}$ is orthogonal to the all the rows of A. rowsp(A) is the space spanned by all the rows of A. so every vector $\bar{a} \in \text{rowsp}(A)$ is just a linear combination of the rows of A. therefore, $\bar{x}$ is orthogonal to any vector $\bar{a} \in \text{rowsp}(A)$

5. $\bar{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \bar{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$|<\bar{u}, \bar{v}>| = 6$$
$$||\bar{u}|| = \sqrt{14}, ||\bar{v}|| = \sqrt{18}$$

And indeed

$$6 < \sqrt{14} * \sqrt{18} = 15.87$$

6.

6.1. Just show: $\overline{v}_j^T \overline{v}_i \begin{cases} = 0 \; if \; i \neq j \\ > 0 \; if \; i = j \end{cases}$

6.2. we are looking for 3 scalars $c_1, c_2, c_3$ that will satisfy: $\overline{u} = c_1\overline{v}_1 + c_2\overline{v}_2 + c_3\overline{v}_3$

We will get 3 equations:

$$x = c_1 + c_2 + 5c_3$$
$$y = c_1 - 3c_2 - c_3$$
$$z = c_1 + 2c_2 - 4c_3$$

And we will get $c_1 = \frac{1}{3}(x + y + z)$, $c_2 = \frac{1}{14}(x - 3y + 2z)$, $c_3 = \frac{1}{42}(5x - y - 4z)$

6.3. $c_1 = 3, c_2 = -4, c_3 = 1$

7. $S$={v1,v2,…,vk} vectors in ℝn

7.1.

Consider the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Our goal is to show that $c_1 = c_2 = \cdots = c_k = 0$.

We compute the dot product of $\mathbf{v}_i$ and the above linear combination for each $i = 1, 2, \ldots, k$:

$$\begin{aligned} 0 &= \mathbf{v}_i \cdot \mathbf{0} \\ &= \mathbf{v}_i \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= c_1\mathbf{v}_i \cdot \mathbf{v}_1 + c_2\mathbf{v}_i \cdot \mathbf{v}_2 + \cdots + c_k\mathbf{v}_i \cdot \mathbf{v}_k. \end{aligned}$$

As $S$ is an orthogonal set, we have $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$.

Hence all terms but the $i$-th one are zero, and thus we have

$$0 = c_i\mathbf{v}_i \cdot \mathbf{v}_i = c_i\|\mathbf{v}_i\|^2.$$

Since $\mathbf{v}_i$ is a nonzero vector, its length $\|\mathbf{v}_i\|$ is nonzero.
It follows that $c_i = 0$.

As this computation holds for every $i = 1, 2, \ldots, k$, we conclude that $c_1 = c_2 = \cdots = c_k = 0$.
Hence the set $S$ is linearly independent.

7.2.

Suppose that $k = n$. Then by part (a), the set $S$ consists of $n$ linearly independent vectors in the dimension $n$ vector space $\mathbb{R}^n$.

Thus, $S$ is also a spanning set of $\mathbb{R}^n$, and hence $S$ is a basis for $\mathbb{R}^n$.

8. From Schaum (example 7.10)

**EXAMPLE 7.10** Apply the Gram–Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace $U$ of $\mathbf{R}^4$ spanned by

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, 2, 4, 5), \qquad v_3 = (1, -3, -4, -2)$$

(1) First set $w_1 = v_1 = (1, 1, 1, 1)$.

(2) Compute

$$v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \frac{12}{4} w_1 = (-2, -1, 1, 2)$$

Set $w_2 = (-2, -1, 1, 2)$.

(3) Compute

$$v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - \frac{(-8)}{4} w_1 - \frac{(-7)}{10} w_2 = \left(\tfrac{8}{5}, -\tfrac{17}{10}, -\tfrac{13}{10}, \tfrac{7}{5}\right)$$

Clear fractions to obtain $w_3 = (-6, -17, -13, 14)$.

Thus, $w_1, w_2, w_3$ form an orthogonal basis for $U$. Normalize these vectors to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ of $U$. We have $\|w_1\|^2 = 4$, $\|w_2\|^2 = 10$, $\|w_3\|^2 = 910$, so

$$u_1 = \frac{1}{2}(1, 1, 1, 1), \qquad u_2 = \frac{1}{\sqrt{10}}(-2, -1, 1, 2), \qquad u_3 = \frac{1}{\sqrt{910}}(16, -17, -13, 14)$$

9. Assume that Q and W are orthogonal matrices. That means that $Q^T Q = I$ and $W^T W = I$. Now let's calculate $(QW)^T QW = W^T Q^T QW = W^T I W = W^T W = I$. Which means that QW is an orthogonal matrix.

11.

- additivity:

Suppose $u, v, w \in V$. Then

$$\begin{aligned} \langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\ &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle. \end{aligned}$$

- homogeneity (scalar multiplication): show by yourselves.

12.

$$\|t \cdot \vec{v}\|^2 = (t\vec{v}, t\vec{v}) = t^2(\vec{v}, \vec{v}) = t^2 \|\vec{v}\|^2$$