Exercise 8

1. Use geometrical terms (and other terms we learned about) to explain the following statement in words:

If the columns of a square matrix A are linearly dependent, then $\det(A) = 0$.

2. Calculate the determinants of the following matrices. You can use the determinant properties we learned in class or the geometrical interpretation of the linear transformation that each matrix represent:

2.1. $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$

2.2. $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2.3. $\begin{pmatrix} 2 & 7 & -2 & 6 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -4 \end{pmatrix}$

2.4. $\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

3. Let A be a square $(n \times n)$ matrix. Prove the following statements:

3.1. If $AA^T = I$, then $\det(A) = \pm 1$.

3.2. If $A = A^2$ and $A \neq I$, then $\det(A) = 0$.

3.3. If A is regular, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

3.4. If A is regular, then A^m is regular (for $m \in \mathbb{N}$).

Note: A square matrix A can be raised to the power of m using the rule:

$$A^m = \underbrace{AAAAA \dots A}_{m\ times}$$

4. Let A be a 2×2 matrix with two linearly dependent columns. Given that $tr(A) = 5$:

4.1. Find $tr(A^2)$

4.2. Find $\det(A^2)$

5. Orthogonal matrices

5.1. Prove that the determinant of an orthogonal matrix is either +1 or -1.