Exercise for chapter 8 – solution

1. If the columns of a square matrix A are linearly dependent, it means that the column space that span the parallelogram after applying A is lower than the original dimension. This means that the volume of the unit parallelogram after applying A is 0.

2.

a. .

By direct computation: $\begin{vmatrix} 4 & -3 \\ 5 & 2 \end{vmatrix} = 4 \cdot 2 - 5 \cdot (-3) = 23$

b. 0*()-1(-1*1+0)+0*()=1

c. .

This is a triangular matrix, so the determinant is the multiplication of the elements on the main diagonal: $\begin{vmatrix} 2 & 7 & -2 & 6 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 2 \cdot (-3) \cdot 1 \cdot (-4) = 24.$

d. .

Switching rows $R_1 \leftrightarrow R_5$, $R_2 \leftrightarrow R_4$ brings about a diagonal matrix, which determinant equals the determinant of the original matrix, since each row flip multiplies the determinant by -1. Thus, $\begin{vmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1) \cdot (-1) \cdot \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} =$

$1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 4.$

3. .

   a. .

      (a) $1 = \det AA^T = \det A \cdot \underbrace{\det A^T}_{=\det A} = (\det A)^2$. *So* $(\det A)^2 = 1$, *and hence* $(\det A) = \pm 1$.

   b. .

      (b) *If we had* $\det A \neq 0$ *then* A *would be invertible and multiplying by its inverse* A^{-1} *would give:* $A^{-1}A^2 = A^{-1}A$, *taht is,* $A = I$, *in contradiction to the assumption.*

   c. .

      Since $AA^{-1} = I$, $|AA^{-1}| = 1$. *This means* $|A| \cdot |A^{-1}| = 1$. *Thus* $|A^{-1}| = \frac{1}{|A|}$. *Note that we can divide by* $|A|$ *because* A *is regular and thus* $|A| \neq 0$

   d. A is regular and therefore $AA^{-1} = I$. Now, $(A^m)(A^m)^{-1} = (AA \ldots A)(AA \ldots A)^{-1} = (AA \ldots A)(A^{-1}A^{-1} \ldots A^{-1}) = (AA \ldots A)I(A^{-1}A^{-1} \ldots A^{-1}) = \cdots = I$. And therefore, A^m is regular too.

4. We can write A *as* $A = \begin{bmatrix} x & ax \\ y & ay \end{bmatrix}$. $tr(A) = x + ay = 5$. *so* $x = 5 - ay$.

$$Therefore: A = \begin{bmatrix} 5 - ay & a(5 - ay) \\ y & ay \end{bmatrix}.$$

$$we\ can\ calculate\ A^2 = \begin{bmatrix} 5(5 - ay) & 5a(5 - ay) \\ 5y & 5ay \end{bmatrix}.$$

   a. $tr(A^2) = 25$
   b. $det(A^2) = 0$ $(the\ columns\ are\ dependent)$.

5. Q is a orthogonal matrix so $Q^TQ = I$. *Therefore* $\det(Q^TQ) = \det(I) = 1$. We know from determinant rules that $\det(Q^TQ) = \det(Q^T)\det(Q) = [det(Q)]^2$. So $[det(Q)]^2 = 1$ *meaning that* $\det(Q) = +1$ *or* -1.