Exercise 9

1. Show that if $\bar{v}_1, \bar{v}_2$ are eigenvectors of A that ae associated with the same eigenvalue λ, then $\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2$ is also an eigenvector of A that is associated with λ.

2. The trace and determinant are closely related to the eigenvalues of matrix.
Let A be a 2×2 matrix, and denote $trace(A) = \tau$ and $\det(A) = \Delta$. Show that in these cases of 2×2 matrices the eigenvalues of A are:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix}$
   3.1. Find the characteristic polynomial of A.
   3.2. Find the eigenvalues of A.
   3.3. Find a basis for the eigenspace of each eigenvalue. How many linearly independent eigenvectors can you find? Is A diagonalizable?

4. Let $A = \begin{pmatrix} -9 & 16 & -4 \\ -8 & 15 & -4 \\ -8 & 16 & -5 \end{pmatrix}$
   4.1. Determine whether $\lambda = -1$ is an eigenvalue of A. If it is, find two linearly independent eigenvectors for this eigenvalue.
   4.2. Show that $(1,1,1)^T$ is an eigenvector of A. What is its eigenvalue?

5. Let A be a singular 3×3 matrix. Assume $\det(2I - A) = \det(-2I - A) = 0$.
   5.1. Is A diagonalizable? If so, write down A in a basis where it is diagonal.
   5.2. What is the dimension of $\ker(A)$?

6. Each of the following real matrices defines some linear transformation on $\mathbb{R}^2$.

$$A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

   For each matrix:
   6.1. Find all real eigenvalues.
   6.2. Find a maximum set S of linearly independent eigenvectors.
   6.3. Determine if the matrix is diagonalizable. If it is, find a change of basis matrix P for which $D = P^{-1}AP$ is diagonal. Verify that D is indeed diagonal.

7. Let $A, B \in M_n(\mathbb{R})$ (i.e, $n \times n$ square matrices with real values). Let λ_1 be an eigenvalue of A and λ_2 be an eigenvalue of B.
Is $\lambda_1\lambda_2$ an always an eigenvalue of AB? It it is, prove. If it isn't, explain in which cases it is and in which cases it is not.

8. *Invariance under change of basis transformation*
In questions 1.1 and 1.2 please note that P is the change of basis matrix to a general basis, not necessarily the eigenbasis (if it even exists).

8.1. Show that the determinant is invariant under change of basis transformation. In other words, given that $B = P^{-1}AP$, show that $det(A) = det(B)$.
8.2. Show that the trace is invariant under change of basis transformation. In other words, show that $trace(A) = trace(B)$ for A, B as defined above.
Guidance: First prove that for any two matrices C, D it is true that $trace(CD) = trace(DC)$, and then use this fact to complete the proof.
8.3. Explain why from (1.1) and (1.2) you can conclude that for any diagonalizable matrix, the eigenvalues sum to the trace, and their product is the determinant.

9. In Ex09, Q3 you studied the matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

Is A diagonalizable? If so, explain and find the change of basis matrix P such that $P^{-1}AP$ is diagonal.

10. True or false? Every diagonalizable matrix is also invertible. Explain.

11. In Q6 you had the following real matrices defines some linear transformation on $\mathbb{R}^2$.

$$A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

Based on your answers there, determine if each matrix is diagonalizable. If it is, find a change of basis matrix P for which $D = P^{-1}AP$ is diagonal. Verify that D is indeed diagonal.

12. Let A be a real symmetric matrix ($A^T = A$).
A is called **positive definite** if for any nonzero vector $\bar{x} \in \mathbb{R}^n$ it is true that:

$$\bar{x}^T A\bar{x} > 0$$

12.1. True or false? All eigenvalues of a positive definite matrix are positive.
12.2. What can you say about the angle between a vector before ($\bar{x}$) and after transformation by a positive definite matrix ($A\bar{x}$)? Note that $\bar{x}$ is a general vector, not necessarily an eigenvector.

13. Let $A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$
13.1. Is A diagonalizable over the real numbers $\mathbb{R}$? Justify your answer.
13.2. Is A diagonalizable over the complex numbers $\mathbb{C}$? Justify your answer.

14. Show that $A = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i & \frac{2}{3}i \\ -\frac{2}{3}i & -\frac{1}{3} - \frac{2}{3}i \end{pmatrix}$ is unitary.

15. True or false?

15.1. All the diagonal entries of a Hermitian matrix are real.

15.2. Every Hermitian matrix is also a normal.

Definition: A complex matrix is said to be normal if it commutes with A^H:

$$AA^H = A^H A$$

A matrix is normal if and only if it is diagonalizable by a unitary matrix U:

$$UAU^{-1} = \Lambda$$