Exercise for chapter 9 – solution

1.

Let $T(v_1) = \lambda v_1, T(v_2) = \lambda v_2$. Then $T((\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) = \alpha_1 \lambda v_1 + \alpha_2 \lambda v_2 = \lambda(\alpha_1 v_1 + \alpha_2 v_2) \rightarrow \alpha_1 v_1 + \alpha_2 v_2$ is an eigen vector of T that belongs to λ.

2.

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\tau = \text{trace}(A) = a + d$$
$$\Delta = \det(A) = ad - bc$$

$$0 \stackrel{!}{=} \det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & c \\ b & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - \lambda d - \lambda a + \lambda^2 - bc = \underbrace{ad - bc}_{\Delta} - \lambda \underbrace{(a+d)}_{\tau} + \lambda^2$$

$$\lambda^2 - \lambda\tau + \Delta = 0$$

$$\boxed{\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix}$

3.1. Find the characteristic polynomial of A.

3.2. Find the eigenvalues of A.

3.3. Find a basis for the eigenspace of each eigenvalue. How many linearly independent eigenvectors can you find? Is A diagonalizable?

3.1 We will calculate the determinant using the 3rd column:

$$det(A - \lambda I) = (3-\lambda)\big((1-\lambda)(-1-\lambda) - 8\big) = (3-\lambda)(-(1-\lambda^2) - 8))$$
$$= (\lambda - 3)(9 - \lambda^2) = (\lambda - 3)(3+\lambda)(3-\lambda)$$

3.2 The eigenvalues of A are therefore $\lambda_1 = 3, \lambda_2 = -3$, where λ_1 has an algebraic multiplicity of 2.

3.3 For $\lambda = -3$:

$$(A - \lambda_2 I)\bar{v} = 0$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 2 & 0 \\ -1 & 1 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

R1<->R3, R3->-R3:

$$\begin{pmatrix} 1 & -1 & -6 \\ 4 & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

R3->R3-R2

$$\begin{pmatrix} 1 & -1 & -6 \\ 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

R2->R2-4R1:

$$\begin{pmatrix} 1 & -1 & -6 \\ 0 & 6 & 24 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

R2 ->R2/6:

$$\begin{pmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We have a free variable. set $z = c$, to get $y = -4c, x = 2c$: $\bar{v} = c\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$.

This vector is the basis for the eigenspace of eigenvalue $\lambda = -3$.

For $\lambda = 3$:

$$(A - \lambda_1 I)\bar{v} = 0$$

$$\begin{pmatrix} -2 & 2 & 0 \\ 4 & -4 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

R1<->R3,

R3->R3- 2R1,

R2->R2+4R1:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

And we get x=y. we can set x=y=a and we also have a free variable. set $z = c$, $\bar{v} = \begin{pmatrix} a \\ a \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent. Thus, this is the basis for the eigenspace of eigenvalue $\lambda = 3$.

We found 3 linearly independent eigenvectors ($\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$). Recall that a matrix is diagonalizable iff it has n independent eigenvectors (where n is the dimension of the matrix). Therefore, A is diagonalizable.

4. .

4.1. If $\lambda = -1$: $Av = -v. so\ Av + v = (A+I)v = 0$. Let's find general solution for this system:

$$(A+I) = \begin{pmatrix} -8 & 16 & -4 \\ -8 & 16 & -4 \\ -8 & 16 & -4 \end{pmatrix}$$

R3->R3- R1, R2->R2- R1:

$$\begin{pmatrix} -8 & 16 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

R1->(-1/8)R1:

$$\begin{pmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There is a solution and therefore, $\lambda = -1$ is an eigenvalue of A.

The general solution is $\bar{v} = \begin{pmatrix} 2s - \frac{1}{2}t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$. $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$ are linearly independent. Thus, this is the basis for the eigenspace of eigenvalue $\lambda = -1$.

4.2. $\begin{pmatrix} -9 & 16 & -4 \\ -8 & 15 & -4 \\ -8 & 16 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue 3.

5.

(6) Let A be a singular 3×3 matrix. Assume $\det(2I - A) = \det(-2I - A) = 0$.

(a) Is A diagonalizable? If so, write down all diagonal matrices similar to A.

(b) What is dimension of the solution space of the system $Ax = 0$?

(a) *Since A is singular, there are non-zero vectors that satisfy $Ax = 0$. This is equivalent to saying that 0 is an eigenvalue of A. Since $\det(2I - A) = 0$, $\lambda = 2$ is an eigenvalue of A. Similarly, from $\det(-2I - A) = 0$ we get that $\lambda = -2$ is an eigenvalue of A. The 3×3 matrix A has therefore 3 distinct eigenvalues, and is thus diagonalizable. Any diagonal matrix with a permutation of $0,\ 2,\ -2$ on its main diagonal is similar to A:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) *The dimension of eigen space of the eigen value 0 is 1, therfore the dimension of the null space of A is 1*

6. Schaum, 9.11 (page 310)

(a) First find $\Delta(t) = t^2 - 3t - 28 = (t - 7)(t + 4)$. The roots $\lambda = 7$ and $\lambda = -4$ are the eigenvalues of A. We find corresponding eigenvectors.

(i) Subtract $\lambda = 7$ down the diagonal of A to obtain

$$M = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}, \quad \text{corresponding to} \quad \begin{array}{r} -2x + 6y = 0 \\ 3x - 9y = 0 \end{array} \quad \text{or} \quad x - 3y = 0$$

Here $v_1 = (3, 1)$ is a nonzero solution.

(ii) Subtract $\lambda = -4$ (or add 4) down the diagonal of A to obtain

$$M = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix}, \quad \text{corresponding to} \quad \begin{array}{r} 9x + 6y = 0 \\ 3x + 2y = 0 \end{array} \quad \text{or} \quad 3x + 2y = 0$$

Here $v_2 = (2, -3)$ is a nonzero solution.

Then $S = \{v_1, v_2\} = \{(3, 1), (2, -3)\}$ is a maximal set of linearly independent eigenvectors. Because S is abasisof $\mathbf{R}^2$, A is diagonalizable. Using the basis S, A is represented by the diagonal matrix $D = \text{diag}(7, -4)$.

(b) First find the characteristic polynomial $\Delta(t) = t^2 + 1$. There are no real roots. Thus B, a real matrix representing a linear transformation on $\mathbf{R}^2$, has no eigenvalues and no eigenvectors. Hence, in particular, B is not diagonalizable.

(c) First find $\Delta(t) = t^2 - 8t + 16 = (t - 4)^2$. Thus, $\lambda = 4$ is the only eigenvalue of C. Subtract $\lambda = 4$ down the diagonal of C to obtain

$$M = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \text{corresponding to} \qquad x - y = 0$$

The homogeneous system has only one independent solution; for example, $x = 1$, $y = 1$. Thus, $\upsilon = (1, 1)$ is an eigenvector of C. Furthermore, as there are no other eigenvalues, the singleton set $S = \{\upsilon\} = \{(1, 1)\}$ is a maximal set of linearly independent eigenvectors of C. Furthermore, because S is not a basis of $\mathbf{R}^2$, C is not diagonalizable.

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7. In the general case this is not true: take the matrices A and C from the previous question (6). 7 is an eigenvalue of A with eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and 4 is an eigenvalue of C with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. AC=$\begin{bmatrix} 31 & 13 \\ 13 & -9 \end{bmatrix}$ and 4*7=28 is not an eigenvalue of this matrix (you can see that by calculate det(AC-28I) which is not 0).

However, in the case that the eigenvectors are the same this is true:

$$Av = \lambda_1 v \,, Bv = \lambda_2 v :$$

$$ABv = A\lambda_2 v = \lambda_2 Av = \lambda_2 \lambda_1 v$$

So $\lambda_2 \lambda_1$ is an eigenvalue of AB.

8.

Solution

1.1 We will solve this question in three method.

Method 1:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \underbrace{=}_{\substack{determinants \\ are\ scalars}} \det(P^{-1})\det(P)\det(A)$$

$$= \det\Big(\underbrace{P^{-1}P}_{I}\Big)\det(A) = \det(A)$$

Method 2:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \underbrace{=}_{\substack{determinants \\ are\ scalars}} \det(P^{-1})\det(P)\det(A)$$

$$= \frac{1}{\det(P)}\det(P)\det(A) = \det(A)$$

Method 3:

$$B = P^{-1}AP \;\rightarrow\; PB = AP \;\rightarrow \det(PB) = \det(AP)$$

$$\det(P)\det(B) = \det(A)\det(P)$$

Since P is invertible $\det(P) \neq 0$, and we can divide both sides by $\det(P)$:

$$\det(B) = \det(A)$$

1.2 We first probe that $tr(AB) = tr(BA)$.

We use the formula of the general term $(AB)_{ij}$ for the case $i = j$:

$$tr(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$$

We can change the order of summation:

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ik}b_{ki}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki}a_{ik}\right) = \sum_{k=1}^{n}(BA)_{ii} = tr(BA)$$

Now, using this proposition we can see that: $tr(B) = tr(P^{-1}AP) = tr(APP^{-1}) = tr(A)$.

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9. We found 3 linearly independent eigenvectors ($\begin{pmatrix}2\\-4\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$).

$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are eigenvector of the eigenvalue $\lambda = 3$. And $\begin{pmatrix}2\\-4\\1\end{pmatrix}$ is an eigenvector of the eigenvalue $\lambda = -3$

Recall that a matrix is diagonalizable iff it has n independent eigenvectors (where n is the dimension of the matrix). Therefore, A is diagonalizable and the change of basis matrix P is the matrix which its columns are the eigenvectors: $P = \begin{pmatrix}1 & 0 & 2\\1 & 0 & -4\\0 & 1 & 1\end{pmatrix}$. And $P^{-1}AP = \begin{pmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & -3\end{pmatrix}$

10. False, we give a counterexample: Consider the 2×2 zero matrix. The zero matrix is a diagonal matrix, and thus it is diagonalizable. However, the zero matrix is not invertible as its determinant is zero.

11. we already answered that in q6.

12.

1. True.

Suppose our matrix A has eigenvalue λ.

If $\lambda = 0$, then there is some eigenvector x so that $Ax = 0$. But then $x^T Ax = 0$, and so A is not positive definite.

If $\lambda < 0$, then there is some eigenvector x so that $Ax = \lambda x$. But then $x^T Ax = \lambda |x|^2$, which is negative since $|x|^2 > 0$ and $\lambda < 0$. Thus A is not positive definite.

And so if A is positive definite, it only has positive eigenvalues.

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2. The dot product of 2 vectors $\bar{v}$ and $\bar{w}$ is: $\bar{v}^T \cdot \bar{w} = |\bar{v}||\bar{w}|cos\theta$. Where θ is the angle between the vectors. So if $\bar{v} = \bar{x}$ and $\bar{w} = A\bar{x}$, θ is the angle between the vector before and after transformation and we get $cos\theta = \frac{\bar{x}^T A\bar{x}}{|\bar{x}||A\bar{x}|}$. The denominator and nominator are both positive so the whole term is positive. Meaning that $cos\theta > 0$.

13.

1. $det(A - \lambda I) = \lambda^2 + 1 = 0$. This eq has no solution over the real numbers and therefore A is not diagonalizable over the real numbers.

2. Over the complex numbers we can solve this eq and get $\lambda = i$ or $\lambda = -i$.

Since A has two distinct eigenvalues over the complex numbers (and A is 2x2 matrix), it must be diagonalizable.

(The following is not part of the question:) the eigenvectors:

For $\lambda_1 = i$: $(A - iI)\overline{v_1} = \begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix}\overline{v_1} = 0$ we get $\overline{v_1} = \begin{pmatrix} \frac{3+i}{2} \\ 1 \end{pmatrix}$

We know that eigenvalues and eigenvectors of real matrices come in conjugate pairs so we can infer that for $\lambda_2 = -i$: $\overline{v_2} = \begin{pmatrix} \frac{3-i}{2} \\ 1 \end{pmatrix}$

14. we can show that in few ways:

1. show that $A^H = A^{-1}$

2. A complex matrix is unitary if and only if its rows (and its columns) form an orthonormal set. We will show that the columns are orthonormal: $\bar{v} = \begin{pmatrix} \frac{1}{3}-\frac{2}{3}i \\ -\frac{2}{3}i \end{pmatrix}$ $and\, \overline{w} = \begin{pmatrix} \frac{2}{3}i \\ -\frac{1}{3}-\frac{2}{3}i \end{pmatrix}$. According to the inner product of complex numbers: $< \overline{v}, \overline{w} >= \bar{v}^{*T}\overline{w} = \frac{2}{3}i\left(\frac{1}{3}+\frac{2}{3}i\right)+\frac{2}{3}i\left(-\frac{1}{3}-\frac{2}{3}i\right) = 0.$ so $v\ and\ w$ are orthogonal. Now we need to show that $|\overline{v}| = 1\ and\ |\overline{w}| = 1$:

$$|\bar{v}| = \sqrt{< \overline{v}, \overline{v} >} = \sqrt{\bar{v}^{*T}\,\overline{v}} = \sqrt{\left(\frac{1}{3}+\frac{2}{3}i\right)\left(\frac{1}{3}-\frac{2}{3}i\right)+\frac{2}{3}i*\left(-\frac{2}{3}i\right)} = \sqrt{\left(\frac{1}{9}+\frac{4}{9}+\frac{4}{9}\right)}$$
$$= 1$$

$$|\overline{w}| = \sqrt{<\overline{w},\overline{w}>} = \sqrt{\overline{w}^{*T}\overline{w}} = \sqrt{-\frac{2}{3}i*\left(\frac{2}{3}i\right)+\left(-\frac{1}{3}+\frac{2}{3}i\right)\left(-\frac{1}{3}-\frac{2}{3}i\right)}$$
$$= \sqrt{\left(\frac{4}{9}-\left(-\frac{1}{9}-\frac{4}{9}\right)\right)} = 1$$

15.

1. True. When you take the transpose, the diagonal entries will not change the position. Then after taking the conjugate, the diagonal entries should be the same. That means they must be real. **Prove it yourself in a rigorous way.**

2. True. If A is Hermitian, so $A^H = A. and\ therefore\ AA^H = A^HA\ because\ AA = AA$.