Linear Algebra – Lesson 1

Vectors, linear combinations, linear dependence

1. General information about the course

2. An introduction to the course

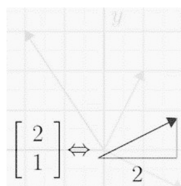
3. Vectors

3.1 Vectors as arrows or lists

A vector is an abstract notion, with many different definition. Two useful definition are these:

- **Vectors as arrows** that start at the **origin**, and have direction and length
- **Vectors as abstract lists**: useful for generalizing the notion of vector (later in the course we will see that this concept is even more abstract: Anything that obeys the rules of summation and multiplication is a vector, even functions)

The two pictures agree with other, and there is a one-to-one correspondence between an arrow and a list of coordinates:



- Notation: $\bar{v}, \bar{u}, v, u, v_x, [v]_x$. Usually, we will use the letter u, v, w to describe vectors.
- In our example, the vector is said to be 2-dimensional ($\bar{v} \in \mathbb{R}^2$) because it has two entries.
- By convention, when we write $\bar{v}$ we mean a **column vector**. Later in the course we will see the row vector as well.

The picture of vectors as arrows is especially useful for building intuition about linear algebra, and we will use it throughout the course.

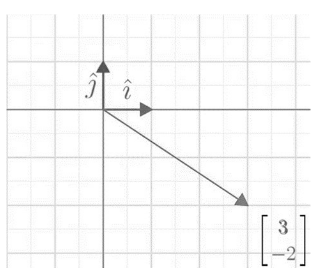
3.2 Basis vectors

Now please imagine the vector $\begin{pmatrix}3\\-2\end{pmatrix}$.

How come you can understand me and translate this list to an image in your mind?

We have a common ground that we agree on – we are thinking about the same coordinate system. In linear algebra, this common ground is called the **basis vectors**.

These are usually denoted $\hat{\imath}, \hat{\jmath}$, where the hat sign $\widehat{\ }$ ("hat") means this is a "unit vector", whose length is 1.



The set of basis vectors defines a "coordinate system".

When we write $\binom{3}{-2}$, we implicitly assume a specific coordinate system, or a specific basis that we're using. The usual basis we're using is called the "standard basis".
This is important, because we could've used a different basis, like the one where the "x" direction is flipped. In this new basis, the same vector will be denoted $\binom{-3}{-2}_{xflip}$.

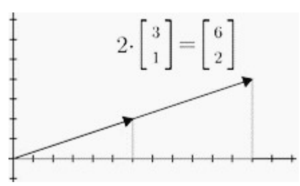
3.3 Vector addition and multiplication

What can we do with vectors? Only two things: Vector addition and scalar multiplication. To explain each of them, we'll start with the numerical "lists" definition, and then talk about its geometrical interpretation.

- **Vector addition**
  Sum each coordinate separately:
  $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$
  The geometrical interpretation is: start with the first vector, then "stick" the second vector at its head. The resulting vectors starts at the origin and ends at the head of the second vector.

  Example: $\binom{3}{-2} + \binom{1}{1} = \binom{4}{-1}$
  Draw this summation.

- **Scalar multiplication**
  Multiplying a vector $\bar{v}$ by a **number** a means multiplying each of its coordinates separately:



  Geometrically, multiplying a vector $\bar{v}$ by a **number** a means changing the length of $\bar{v}$ by a factor of a.
  - $2\bar{v} = \bar{v} + \bar{v}$, which makes sense: it's simply taking twice the same vector
  - $-\bar{v}$ means flipping the direction of the vector (it's called the negative of $\bar{v}$)

  This is called "scalar multiplication". A scalar is just another name for a number. This name makes sense, because the scalar **scales** the vector.

3.4 Linear combinations

A linear combination of vectors is any combination of them, using vector addition and scalar multiplication.
A linear combination of $\bar{u}, \bar{v}$ is defined like this:
$$\bar{w} = a\bar{u} + b\bar{v} \qquad (a, b \in \mathbb{R})$$
The result is a vector of the same dimension (same size).

Example

$\bar{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ $\bar{v} = \begin{bmatrix}2\\1\\3\end{bmatrix}$. So $2\bar{u} - 3\bar{v} = \begin{bmatrix}2\\3\\2\end{bmatrix} - \begin{bmatrix}6\\3\\9\end{bmatrix} = \begin{bmatrix}-4\\0\\-7\end{bmatrix}$

The zero vector of any dimension is simply a vectors of zeros, usually denoted by 0. So for any $\bar{v}$

$$\bar{v} + 0 = \bar{v}.$$

Example

Write the vector $\bar{v} = \begin{bmatrix}3\\2\end{bmatrix}$ as a linear combination of two vectors.

The simplest solution (after $\bar{v} + 0$) is $\bar{v} = 3\hat{\imath} + 2\hat{\jmath}$. In other words, any vector can be written as a linear combination of the basis vectors.

Example

But what if I asked you to use some specific vectors which are not the standard basis vectors?

Write the vector $\bar{v} = \begin{bmatrix}3\\2\end{bmatrix}$ as a linear combination of the vectors $\bar{u}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \bar{u}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

We are looking for two scalars c_1, c_2 that will satisfy:

$$\bar{v} = c_1\bar{u}_1 + c_2\bar{u}_2$$

$$\begin{bmatrix}3\\2\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$$

Each term will give us one equation:

$$3 = c_1 - c_2$$
$$2 = c_1 + c_2$$

By summing the equations, we get:

$c_1 = 2.5$

$c_2 = -0.5$

So, linear combinations allow you to take a set of vectors, and create new vectors by using vector addition and scalar multiplication.

Question: How many new vectors can you create using linear combinations?

Answer: Infinite vectors (unless the set of vectors is the 0 vector).

3.5 The span

Let's say that we have some vectors. Now we can ask: what is the set of all vectors that you can get from a linear combination of these specific vectors? The answer is "the span of these vectors".

Definition:

Let V be a set of vectors $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$. The span of V consists of all linear combinations of vectors in V. In other words, the span of V consists of all vectors of the form:

$$a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n \quad (a_i \in \mathbb{R})$$

Let's see an example.

Span – Example 1: What is the span of the vectors $\bar{v} = \begin{bmatrix}1\\0\end{bmatrix}, \bar{u} = \begin{bmatrix}1\\1\end{bmatrix}$?

As you can see, we can actually get to any other vector in the plane, by taking a linear combination of $\bar{v}$ and $\bar{u}$. In other words, the span of $\bar{v}$ and $\bar{u}$ is $\mathbb{R}^2$, the whole plane.

Definition: Let K be a set of n vectors $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$. Then $Sp(K)$ ("the span of K") is the set of all the linear combinations of the vectors in K:

$$Sp(K) = \{\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots + \alpha_n\bar{v}_n | \alpha_i \in R, n \in \mathbb{N}\}$$

Or equivalently:

$$Sp(K) = \left\{\sum_{i=1}^{n} \alpha_i\bar{v}_i \, | \alpha_i \in R, n \in \mathbb{N}\right\}$$

Question 1: Geometrically, what is the span of a single vector?
Answer 1: It's a line.

Question 2: What is the span of two vectors?
Answer 2: It could be a plane, but in special cases, it could actually be a line. This will happen if one vector is just the scalar multiplication of the other: $\bar{u} = a\bar{v}$. In that case, we could never "leave" the line.

Example: You expect two vectors $\bar{v} = \begin{pmatrix}1\\2\end{pmatrix}, \bar{u} = \begin{pmatrix}-2\\-4\end{pmatrix}$ to span the entire 2D plane ($\mathbb{R}^2$), but in fact, they only span a **subspace** of the plane – a single line.

This leads us to a very important concept in linear algebra – linear dependence and independence.

3.6 Linear dependence and independence

Given a **set** of vectors, we can ask if they are "linearly dependent" or "linearly independent".

Two equivalent definitions of **linear dependency** for a set of vectors:

- One of more of the vectors can be expressed as a linear combination of the remaining vectors
- One or more of the vectors is inside the span of the remaining vectors

Note: Linear dependence is a property of the set! (For example, an entire set of vectors can be linearly dependent simply because two of its vectors are dependent on each other).

Example: Let $\bar{v} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \bar{u} = \begin{pmatrix}1\\0\\2\end{pmatrix}, \bar{w} = \begin{pmatrix}-2\\1\\-4\end{pmatrix}$. Are they linearly dependent?

One way to answer this is to try to express each vector as the linear combination of the other two.

$\bar{w} = a\bar{v} + b\bar{u}$

$$\begin{pmatrix}-2\\1\\-4\end{pmatrix} = a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\2\end{pmatrix}$$

This gives us three equations, one for each entry:

$-2 = 0\text{a} + 1\text{b} \quad \rightarrow \quad b = -2$

$1 = 1a + 0b \quad \rightarrow \quad a = 1$

$-4 = 1a + 2b \quad \rightarrow \quad -3 = 1 - 4$

Is this enough to say that they are linearly independent? No, because we have to show that any vector in this set cannot be expressed as a linear combination of the rest of the vectors.

Given 3 vectors, $\bar{v}, \bar{u}, \bar{w}$, they are called **linearly independent** if and only if none of them is a linear combination of the others:

$\bar{v} \neq a\bar{u} + b\bar{w} \quad for\ any\ a, b \in \mathbb{R}$

$\bar{u} \neq a\bar{v} + b\bar{w} \quad for\ any\ a, b \in \mathbb{R}$

$\bar{w} \neq a\bar{u} + b\bar{v} \quad for\ any\ a, b \in \mathbb{R}$

This is equivalent to saying that:

$a\bar{v} + b\bar{u} + c\bar{w} = 0 \quad iff \quad a = b = c = 0$

Question: Can you see why?

This leads us to the formal definition of linear independency:

Definition: a set K of n vectors is called linearly independent iff:

$$\sum_{i=1}^{n} a_i \bar{v}_n = 0 \qquad iff \quad a_1 = a_2 = \cdots = a_n = 0$$

3.7 Dimension

While we haven't yet defined the notion "vector space", we will define the **dimension** of a vector space: The dimension of a vector space is the maximal number of linearly independent vectors that you can choose from this space.