Linear Algebra – Lesson 4

Matrices and sets of linear equations

Today we will talk about one of the fundamental problems of linear algebra – solving systems of linear equations.

Matrix equation as a set of linear equations

So far in the course we've talked about vectors and matrices from a geometrical point of view. Matrices were a way to represent linear transformation that map one vector to another, or one space to another.

Now we will see how we can use vectors and matrices as tools for solving problems. Specifically, we will solve systems of linear equation.

1. A linear equation

A linear equation in unknowns $x_1, . x_2, \dots, x_n$ is an equation that can be written as:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Or:

$$\sum_{i=1}^{n} a_ix_i = b$$

Where the constant a_i is called the coefficient of x_i, and b is called the constant term.

Usually, we are interested in finding specific values for $\{x_i\}_{i=1}^{n}$ that satisfy the equation.

Notice that by definition, if an equation is linear in $x_1, \dots, x_n$, it doesn't include terms like x_i^2, x_ix_j etc.

2. A system of linear equations

A system of linear equations is a set of linear equations with the same unknowns. For example:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Here, a_{ij} is the coefficient of the j^{th} variable in the i^{th} equation.

For example:

$$2x + 3y - 4z = 7$$
$$x - 2y - 5z = 3$$
$$-x + 2z = 4$$

In general, we can solve such a system (i.e., find x,y,z that satisfy the equations) by multiplying the equations by scalars, or by adding/subtracting the equations from one another.

3.1 Matrix form of the system of linear equations

A system of linear equations be written in matrix form. For the above example:

$$\underbrace{\begin{pmatrix} 2 & 3 & -4 \\ 1 & -2 & -5 \\ -1 & 0 & 2 \end{pmatrix}}_{\substack{coefficient \\ matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}$$

Or generally:

$$A\bar{x} = b$$

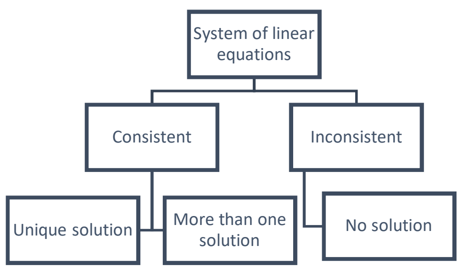
A is called the coefficient matrix. Here $\bar{x}$ is no longer a variable, but a vector of variables: $\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Solving a system of linear equations is equivalent to solving a matrix equation. We are looking for a vector $\bar{x}$ that will satisfy the equation. Such a vector is called a particular solution to the system. The set of all such possible vectors is called the general solution or the solution set of the system.

Why is this useful? It allows us to use tools from linear algebra to study the system, characterize it, and solve it. We start by discussing whether a solution even exists, and building some intuition about it.

3.2 A unique solution doesn't always exist

It turns out that not every set of linear equations has a unique solution. In fact, there is a theorem in linear algebra saying that given a set of linear equations, there are only three possible scenarios[1]:



3.3 How come these are the three possible scenarios? The row picture

We want to get some intuition on what linear systems of equations "look like", and then use this intuition to understand why only these 3 cases are possible – unique solution, infinite solutions or no solution at all.

There are two equivalent ways to think of matrix equations – the row-picture and the column-picture[2]. We will start with the row-picture in 2D, and then go to the column-picture in 2D. Then we'll do the same for 3D.

[1] Actually, this is true only if the matrix is defined over an infinite field K, like $\mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers).

[2] The following examples are taken from Gilbert Strang's lectures on linear algebra in MIT. In fact, we also assume that not both a_i, b_i are zero.

3.3.1 The row picture in 2D

Let's look at this two dimensional example:

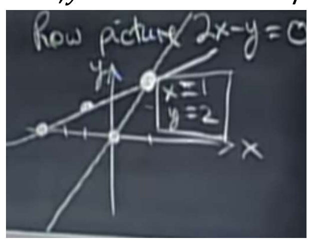
$$2x - y = 0$$
$$-x + 2y = 3$$

To remember this is a linear algebra course, we'll also write it in matrix form:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Plugging the first equation ($y = 2x$) in the second equation, we can find y and the find x. But let's see how to think about it graphically.

Each one of these equations represents a relation between x and y. Specifically in 2D, this relation is a line. For each line, there are many pairs of x,y values that satisfy the equation, those that lie on the line:

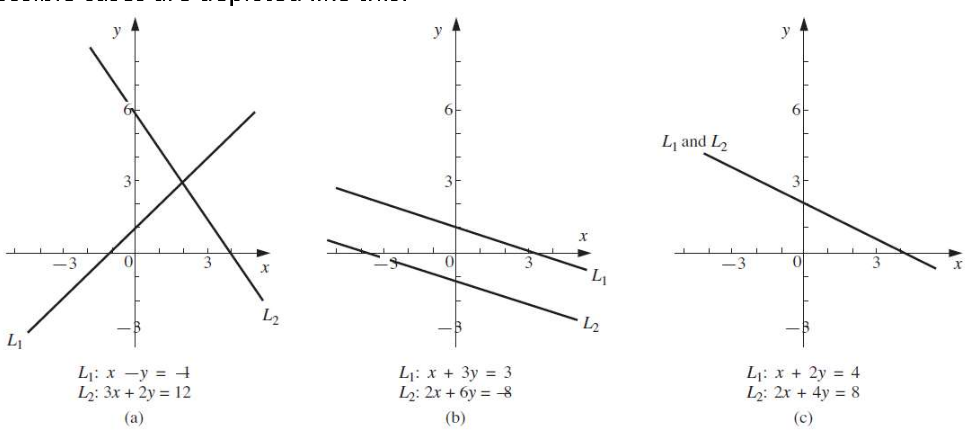


A solution to both equation is a point that lies on both lines. In this example, you can see that this is: $x = 1, y = 2$.

Question: What might go wrong to make it so there is no solution?
Answer: If the two lines don't share any point, we will have no solution at all.

Question: What might cause a case of infinite solutions?
Answer: If the two lines merge, any point on one of them is also a point on the other, and we get an infinite number of solutions.

The three possible cases are depicted like this:



3.3.2 The column picture in 2D

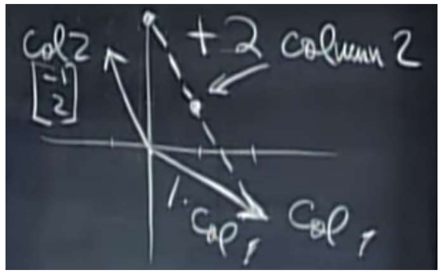
In the row picture, we looked at every row separately. This was the natural way to go because each row is a separate equation. However, the matrix form of the equation calls for a second picture, the column picture.

Let's rewrite our system like this:

$$x\begin{pmatrix}2\\-1\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

We are still looking for x and y, but now we are actually thinking about it as finding the linear combination of the matrix columns that will give us the right answer (this is what we saw in lesson 2: a matrix-vector product is nothing more than a linear combination of the matrix columns).

You already know the solution from the row picture, so let's test if it works:



Question: Could we find a solution for any $\bar{b}$, or was there something special about $\bar{b}=\begin{pmatrix}0\\3\end{pmatrix}$?

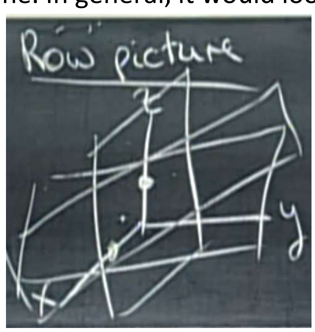
Answer: This is actually like asking "What are all the possible linear combinations of the columns of the matrix?". In other words, it is like asking "What is the span of the matrix columns?". In this case, we can see that the columns span the entire plane, so indeed we could find a solution for any $\bar{b}$. This means that there is something fundamental within the matrix itself (leaving the $\bar{b}$ vector aside) that gives us insight about the problem.

3.3.3 The row picture in 3D

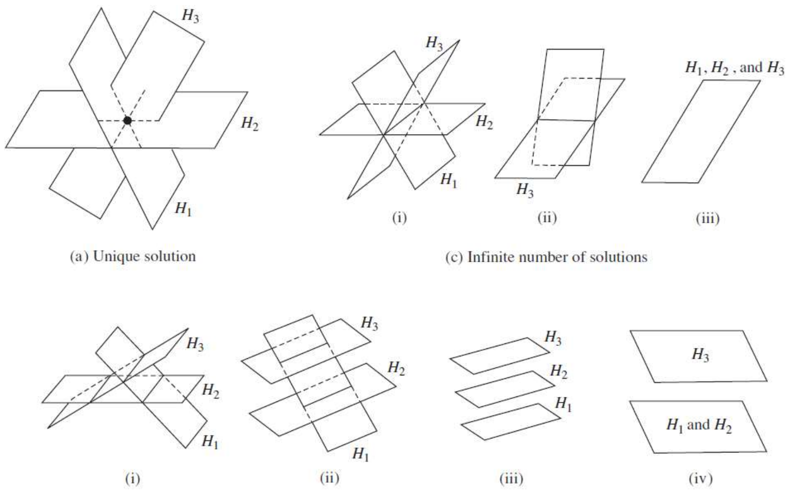
Now let's move on to 3D examples, and look at the following system to explore it from the row picture point of view:

$$2x-y=0$$
$$-x+2y-z=-1$$
$$-3y+4z=4$$

Each of these equations describes a plane. In general, it would look something like this:



So the solution to the problem is some point in space that is shared by all these planes. Let's not bother ourselves with actually finding this point. Maybe you can already appreciate that this row pictures becomes more and more complicated as the dimension of the problem increases. The three possible cases we have for the problem are depicted like this:



(b) No solutions

3.3.4 The column picture in 3D

Remember that in the column picture, we are looking for a linear combination of the columns, that will produce the desired $\bar{b}$:

$$x\begin{pmatrix}2\\-1\\0\end{pmatrix}+y\begin{pmatrix}-1\\2\\-3\end{pmatrix}+z\begin{pmatrix}0\\-1\\4\end{pmatrix}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$$

Here you can easily see that $x = 0, y = 0, z = 1$ solves the problem.

Again we could ask, do we have a solution for any $\bar{b}$? In other words, do linear combination of the columns fill the 3D space? Do they span $\mathbb{R}^3$?

The answer will be “no” only in those special cases for which the columns do not span the entire space. What does this tell us about linear dependence between the columns? If they are linearly dependent (e.g., one vector actually lies on the plane spanned by the other two), then all linear combinations of the matrix columns will always be constrained to lie within this plane, so any $\bar{b}$ outside this plane will be unreachable.

Conclusion

The column picture point of view tells us that sometimes it is helpful to think about the result of $A\bar{x}$ is a linear combination of the columns of A.

4. Using tools of linear algebra to solve linear system of equations

Let’s see how we can use the linear algebra terms we’ve already learned to speak about linear systems of equations. Again, we are looking for a vector $\bar{x}$ that satisfies:

$$A\bar{x} = \bar{b}$$

Now, let’s say that A is a square regular matrix (i.e., it is invertible). Then A^{-1} exists. If we multiply both sides of the equation by A^{-1} from the left, we get:

$$A^{-1}A\bar{x} = A^{-1}\bar{b}$$
$$\bar{x} = A^{-1}\bar{b}$$

In other words, finding a unique solution to the system is equivalent to finding the inverse matrix A^{-1}.

How can we tell which case we're dealing with?
First, we find the determinant of the coefficient matrix A:

- If $\det(A) \neq 0$, then the matrix A is invertible (A^{-1} exists), and the system of equations has a unique solution: $x = A^{-1}b$.
- If $\det(A) = 0$, the matrix A is not invertible. The system of equations either has no solution at all, or it has infinite solutions.

Question: How does the determinant rule relate to the column picture we saw before?
Answer: A determinant of 0 means that the span of the matrix columns is smaller than the dimension of the columns (remember, it means that the volume of a paralleloid is 0). In this case, either we're lucky and the subspace that they span has infinite solutions, or we're unlucky, and $\bar{b}$ resides in the subspace outside the one spanned by the matrix columns, so no solution exists.

Note: If there is more than one solution, then necessarily there are infinite solution. You will prove this in the exercise.

5. Homogeneous systems and inhomogeneous systems
In general, we make a distinction between two types of linear system: homogeneous and inhomogeneous. In a homogeneous system, all the constant terms are 0, and it has the form:

$$A\bar{x} = 0$$

(remember, by 0 we mean here the zero vector).

In an inhomogeneous system, at least one entry of $\bar{b}$ is non-zero, and it has the form:

$$A\bar{x} = \bar{b} \quad (\bar{b} \neq 0)$$

5.1 Solving a homogeneous system
Notice that $\bar{x} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}$ is always a solution to a homogeneous system. This is called "the trivial solution".
We are interested in the non-trivial solution. We will use the following important theorem:

Theorem: A homogeneous system $A\bar{x} = 0$ with more unknowns than equations has a non-zero solution[3].

Example – homogeneous system with a non-trivial solution
Consider the following system:

$$x + y - z = 0$$
$$2x - 3y + z = 0$$
$$x - 4y + 2z = 0$$

We can eliminate x in the second and third equations:

[3] This comes from the fact that in row echelon form (see "Gaussian elimination"), if the number of equations (r) is smaller than the number of unknowns (n), $r < n$, then the system has a nonzero solution.

$$x + y - z = 0$$
$$-5y + 3z = 0$$
$$-5y + 3z = 0$$

But since the last two equations are the same:

$$x + y - z = 0$$
$$-5y + 3z = 0$$

Now, we can express both x and y as a function of z:

$$x + y = z$$
$$y = \frac{3}{5}z$$

Or:

$$x = \frac{2}{5}z$$
$$y = \frac{3}{5}z$$

We call z the independent (or free) variable, while x, y are the dependent variables (because we describe them using z).

Sometimes people like to describe the solution in "parametric form". For example, we can replace the free-variable z with a parameter a to remind ourselves that it can be assigned any value. The general parametric solution then becomes:

$$(x, y, z) = \left(\frac{2}{5}a, \frac{3}{5}a, a\right) = a\left(\frac{2}{5}, \frac{3}{5}, 1\right)$$

We say that the solution set of this system is one-dimensional, since it can be expressed a linear combination of one vector (which we can scale as we wish using the parameter a).

If you are looking for a particular solution, choose some arbitrary value for a, for example:

$$(x_0, y_0, z_0) = (2,3,5)$$

5.2 A basis for the solution set of a homogeneous system

Let's define the term "basis for the general solution" more precisely:

Let W denote the general solution of a homogeneous system $A\bar{x} = 0$. That is, any vector $\bar{w} \in W$ is a solution to the system.

A list of nonzero solution vectors $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_s$ is called a basis for W if each solution $\bar{w} \in W$ can be expressed uniquely as a linear combination of the vectors $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_s$:

$$\bar{w} = a_1\bar{u}_1 + a_2\bar{u}_2 + \cdots + a_s\bar{u}_s$$

The number s of such basis vectors is called the dimension of W.

Example

Suppose we started with some system of equations and finally obtained these two equations:

$$x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 = 0$$
$$x_3 - 3x_4 + 2x_5 = 0$$

By setting x_2, x_4, x_5 as parameters a, b, c, we get:

$x_3 = 3b - 2c$
$x_1 = 4c - 2b + 3(3b - 2c) - 2a = -2a + 7b - 2c$

So that the general solution is:

$$(x_1, x_2, x_3, x_4, x_5) = (-2a + 7b - 2c, \quad a, \quad 3b - 2c, \quad b, \quad c)$$

Notice that we can look at each parameter separately and write this general solution as a linear combination:

$$a(-2,1,0,0,0) + b(7,0,3,1,0) + c(-2,0,-2,0,1)$$

This means that one possible basis for the general solution set is these three vectors:

$$\{(-2,1,0,0,0), \quad (7,0,3,1,0), \quad (-2,0,-2,0,1)\}$$

And the dimension of the solution space is $\dim(W) = 3$. This makes sense, because we started with 5 variables, and 2 independent equations. The two equations allowed us to express two of the variables as a function of the remaining three.

6. Gaussian elimination

In the previous lesson we talked about a general algorithm for finding the inverse matrix A^{-1}, called Gaussian elimination. A very similar form of elimination is used to solve linear systems of equations systematically.

Gaussian elimination is based on the two following theorems:

- Two systems of linear equations have the same solutions iff each equation in each system is a linear combination of the equations in the other system. Two such systems are then called equivalent systems.
- Suppose a system $\mathcal{M}$ of linear equations is obtained from a system $\mathcal{L}$ of linear equations by a finite sequence of elementary operations. Then $\mathcal{M}$ and $\mathcal{L}$ have the same solutions.

The idea is then to use Gaussian elimination to obtain a system of linear equation whose solution is easy to find.

If Gaussian elimination succeeds, it gives an easy solution to the system. If it fails, it tells us that the system has no solution.

6.1 Gaussian elimination example (from Schaum, page 68)

Consider the following system:

$$x - 3y - 2z = 6$$
$$2x - 4y - 3z = 8$$
$$-3x + 6y + 8z = -5$$

Or in matrix form, $A\bar{x} = \bar{b}$:

$$\begin{pmatrix} 1 & -3 & -2 \\ 2 & -4 & -3 \\ -3 & 6 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -5 \end{pmatrix}$$

Our purpose is to bring the matrix to a form in which the values below the diagonal are 0[4], using two elementary operations:

- Multiplication of a row by a number
- Subtraction of one row from the other
- (and possibly, also changing the order of rows, but be careful with that)

$$\begin{pmatrix} 1 & -3 & -2 \\ 2 & -4 & -3 \\ -3 & 6 & 8 \end{pmatrix} \begin{matrix} R2 \to R2 - 2R1 \\ R3 \to R3 + 3R1 \end{matrix} \to \begin{pmatrix} 1 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$

We are done with row 1. Now, eliminate the first non-zero value in row 3:

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & -3 & 2 \end{pmatrix} R3 \to R3 + \frac{3}{2} R2 \to \begin{pmatrix} 1 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3.5 \end{pmatrix}$$

The leading non-zero value in each row is called a pivot. Now, if we apply the same process to the vector $\bar{b}$, we get:

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3.5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$$

The fact that we found three non-zero pivots tells us that the system has a unique solution. Can you see how to easily find this solution?

Question: In the process of Gaussian elimination, under what circumstances will you determine that there are infinite solutions? And no solutions at all?

Notes

- You are encouraged to watch the video in the link on the Moodle website (changing the speed to x1.5 may be useful): https://www.youtube.com/watch?v=2GKESu5atVQ
- Gaussian elimination can be used also for non-square matrices.
- Gaussian elimination is useful not only for solving linear equations. Later in the course, we will see that Gaussian elimination is useful for finding a basis for the row space of a matrix, and a basis for the column space of a matrix.

7. Inhomogeneous linear systems

7.1 The solution set of an inhomogeneous system

The solution set of an inhomogeneous system is closely related to the solution of a very similar homogeneous system.

[4] Such a matrix is called "upper triangular", and is often denoted by U.

Let $A\bar{x} = \bar{b}$ be an inhomogeneous system of linear equations. $A\bar{x} = 0$ is called the associated homogeneous system.

Now, let $\bar{v}_0$ be a particular solution of $A\bar{x} = \bar{b}$, and let W be the general solution of $A\bar{x} = 0$. Then the general solution of $A\bar{x} = \bar{b}$ is:

$$U = \bar{v}_0 + W = \{\bar{v}_0 + \bar{w} | \, \bar{w} \in W\}$$

Why is this a solution to the system? Since any solution to the homogeneous system doesn't affect $\bar{v}_0$:

$$A(\bar{v}_0 + \bar{w}) = A\bar{v}_0 + A\bar{w} = \bar{b} + 0 = \bar{b}$$

In fact, there's one more step to complete the proof:

3.28. Prove Theorem 3.15. Let v_0 be a particular solution of $AX = B$, and let W be the general solution of $AX = 0$. Then $U = v_0 + W = \{v_0 + w : w \in W\}$ is the general solution of $AX = B$.

Let w be a solution of $AX = 0$. Then

$$A(v_0 + w) = Av_0 + Aw = B + 0 = B$$

Thus, the sum $v_0 + w$ is a solution of $AX = B$. On the other hand, suppose v is also a solution of $AX = B$. Then

$$A(v - v_0) = Av - Av_0 = B - B = 0$$

Therefore, $v - v_0$ belongs to W. Because $v = v_0 + (v - v_0)$, we find that any solution of $AX = B$ can be obtained by adding a solution of $AX = 0$ to a solution of $AX = B$. Thus, the theorem is proved.

8. Dependence on parameters

Given a linear system of equations, we can use everything we learned in this lesson to determine whether the system has a unique solution, infinite solutions or no solution at all.

As you will see in the exercise, if the system of equations has some parameter a, the question of the number of solutions depend on the value of a.

Additional resources

The next several videos may be helpful for clarifying the concepts we discussed in this lesson:

- A visualization of equations in three variables as planes:
  Algebra 42 - Visualizing Linear Equations in Three Variables
- The eight possible scenarios in the row picture point of view for a 3x3 system:
  Algebra 43 - Types of Linear Systems in Three Variables
- The same concept of parametric solution sets can be used without Gaussian elimination, as shown in Algebra 47 - Describing Infinite Solution Sets Parametrically.
- Algebra 56 - A Geometrical View of Gauss-Jordan Elimination
- Algebra 60 - Parametric Equations with Gauss-Jordan Elimination