Linear Algebra – Lesson 12
Some eigenvalues proofs

1.1 Theorem I
Let $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ be eigenvectors of a matrix A associated with distinct eigenvalues $\lambda_1, . \lambda_2, \dots, \lambda_n$. Then $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ are linearly independent.

Proof
Let's take a look at two of the eigenvectors $\bar{v}_i, \bar{v}_j$:

$$A\bar{v}_i = \lambda_i \bar{v}_i$$
$$A\bar{v}_j = \lambda_j \bar{v}_j$$
$$\lambda_i \neq \lambda_j$$

Assume that the two eigenvectors are linearly dependent. Then we can express one as a scalar multiplication of the other:

$$\bar{v}_i = k\bar{v}_j$$

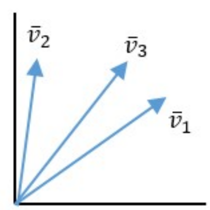
But this means that:

$$A\bar{v}_i = A(k\bar{v}_j) = kA\bar{v}_j = k\lambda_j\bar{v}_j = \lambda_j\bar{v}_i$$

But this is impossible, since $A\bar{v}_i = \lambda_i\bar{v}_i$, and $\lambda_i \neq \lambda_j$. This completes the proof.

But wait! This proof is deceiving – it's actually completely wrong. Can you spot the flaw here?

The point is this: we only proved that eigenvectors of distinct eigenvalues cannot be linear multiples of each other. But linear dependence is a property of a set of vectors: we must prove that any eigenvector cannot be expressed as a linear combinations of all other vectors. For example, our "proof" does not take into consideration a possible scenario:



While $\bar{v}_3$ is not linearly dependent on either $\bar{v}_1$ or $\bar{v}_2$ separately, it is linearly dependent on both of them together. The set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly dependent. A correct proof must take this into account.

(Correct) Proof
Let's suppose that the theorem is false. We assume that one vector ($\bar{v}_1$) can be expressed as a linear combination of the rest of the eigenvectors ($\bar{v}_2, \dots, \bar{v}_n$, which are linearly independent):

$$\bar{v}_1 = \sum_{i=2}^{n} a_i\bar{v}_i = a_2\bar{v}_2 + a_3\bar{v}_3 + \cdots + a_n\bar{v}_n$$

Since eigenvectors are nonzero vectors, at least one of the coefficients is nonzero (some $a_k \neq 0$).
Multiply both sides by A:

$$A\bar{v}_1 = a_2A\bar{v}_2 + a_3A\bar{v}_3 + \cdots + a_nA\bar{v}_n$$
$$\lambda_1\bar{v}_1 = \lambda_2a_2\bar{v}_2 + \lambda_3a_3\bar{v}_3 + \cdots + \lambda_na_n\bar{v}_n$$

We want to get some contradiction, so we multiply the first equation by λ_1:

$$\lambda_1 \bar{v}_1 = \lambda_1 a_2 \bar{v}_2 + \lambda_1 a_3 \bar{v}_3 + \cdots + \lambda_1 a_n \bar{v}_n$$

By subtracting the last two equations:

$$0 = (\lambda_1 - \lambda_2) a_2 \bar{v}_2 + (\lambda_1 - \lambda_3) a_3 \bar{v}_3 + \cdots + (\lambda_1 - \lambda_n) a_n \bar{v}_n$$

Since $\bar{v}_2, \ldots, \bar{v}_n$ are linearly independent, this means that all the coefficients are zero:

$$(\lambda_1 - \lambda_2) a_2 = 0$$
$$(\lambda_1 - \lambda_3) a_3 = 0$$
$$\ldots$$
$$(\lambda_1 - \lambda_n) a_n = 0$$

But since the eigenvalues are distinct, we get that:

$$a_2 = 0, a_3 = 0, \ldots, a_n = 0$$

However, this means that $\bar{v}_1 = 0$, and that's impossible, since an eigenvector cannot be the zero vector.

1.2 Theorm 2

For a real symmetric matrix A, eigenvectors of distinct eigenvalues are orthogonal.

Proof

Let $\bar{x}$ and $\bar{y}$ be two eigenvectors corresponding to the eigenvalues $\lambda_1 \neq \lambda_2$:

$$A\bar{x} = \lambda_1 \bar{x}$$
$$A\bar{y} = \lambda_2 \bar{y}$$

Since we want to get $\bar{x}^T \bar{y} = \bar{y}^T \bar{x} = 0$, we will multiply each equation by the relevant vector from the left:

$$\bar{y}^T A \bar{x} = \lambda_1 \bar{y}^T \bar{x}$$
$$\bar{x}^T A \bar{y} = \lambda_2 \bar{x}^T \bar{y}$$

Now, take the transpose of the second equation:

$$\bar{y}^T A \bar{x} = \lambda_1 \bar{y}^T \bar{x}$$
$$\bar{y}^T A^T \bar{x} = \lambda_2 \bar{y}^T \bar{x}$$

Using $A^T = A$ (because A is symmetric), we get that both right hand sides are equal:

$$\lambda_1 \bar{y}^T \bar{x} = \lambda_2 \bar{y}^T \bar{x}$$
$$(\lambda_1 - \lambda_2) \bar{y}^T \bar{x} = 0$$

Since $\lambda_1 \neq \lambda_2$, we must have:

$$\bar{y}^T \bar{x} = 0$$

As we wanted.

Notice that if A has n distinct eigenvalues, this theorem tells us that it has an (orthogonal) eigenbasis. This still doesn't prove that any symmetric matrix Is diagonalizable, because we haven't treated the case of repeated eigenvalues (those with algebraic multiplicity greater than 1). You can read a proof of this latter case here (proof 2).