Some additional questions – solution

1. If I is the matrix representation in the "standard" basis, the matrix representation in another basis is $S^{-1}IS$, where S is the matrix corresponding to the new basis. Now, $S^{-1}IS = S^{-1}S = I$. Thus I is invariant under change of basis transformation.

2. Let's guess that $(I-A)^{-1} = I + A + A^2 + \cdots + A^{m-1}$ and show that it satisfies the definition of the inverse matrix : $(I-A)(I + A + A^2 + \cdots + A^{m-1}) = I + A + A^2 + \cdots + A^{m-1} - A(I + A + A^2 + \cdots + A^{m-1}) = I + A + A^2 + \cdots + A^{m-1} - A - A^2 - \cdots - A^{m-1} - A^m = I - A^m = I$. The last equality is because A is nilpotent with nilpotency index m. we got that $(I-A)(I + A + A^2 + \cdots + A^{m-1}) = I$. Thus, $(I-A)^{-1} = I + A + A^2 + \cdots + A^{m-1}$.

3. If $\bar{u}$ and $\bar{v}$ are distinct solutions, then $A\bar{u}$= b and $A\bar{v}$ =b, so $A\bar{u}$= $A\bar{v}$ and therefore $A(\bar{u}-\bar{v})$=0 with $\bar{w}$=$\bar{u}-\bar{v}$≠0. Now we can take any scalar k and show that $\bar{u}+k\bar{w}$ is a solution of our linear system. Indeed $A(\bar{u}+k\bar{w})$=$A\bar{u}+kA\bar{w}$=b+0=b and no two column vectors $\bar{u} + k_1\bar{w}$ and $\bar{u} + k_2\bar{w}$, with $k_1 \neq k_2$ are equal. (Indeed, from $\bar{u} + k_1\bar{w} = \bar{u} + k_2\bar{w}$ we can derive $(k_1 - k_2)\bar{w}$=0; as $\bar{w}$≠0, we get $k_1 - k_2 = 0 \rightarrow k_1 = k_2$). Since we have infinitely many scalars available, the system has infinitely many solutions.