Linear Algebra – Lesson 14

Course review

Review questions

1. **Inferring about $\ker(A)$** [ex12, 2006]

   Let T be a linear transformation $T\bar{v} = A\bar{v}$, where A is a 4×4 matrix. The characteristic polynomial of A is $(\lambda^2 - 1)(\lambda^2 - 3\lambda - 10)$. Is this enough information to determine $\ker(T)$? If it is, find $\ker(T)$. If it is not, explain why or give a counter example.

   <u>Solution</u>

   A is the coordinate based representation of T. In order for T to have a nontrivial kernel, $A\bar{x} = 0$ should have non-trivial solutions. This is equivalent to saying that 0 is an eigenvalue of A.
   We test if 0 is an eigenvalue of A (i.e., if it is a root of the characteristic polynomial):
   $$(0^2 - 1)(0^2 - 3 \cdot 0 - 10) = 10 \neq 0$$
   Since 0 is not a root of the characteristic polynomial, $\ker(T) = \{0\}$.

2. **Vector subspace**

   Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 | x + y = 0, y - 2z = 0 \right\}$.
   Is W a vector space?

   <u>Solution I</u>

   Yes it is. Since $W \subset \mathbb{R}^3$, and $\mathbb{R}^3$ is a vector space, it is enough to test whether W is a vector subspace (i.e., if it is closed under linear combinations).
   First, notice that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$. Now, test for vector addition. Assume $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in W$:
   $$\begin{cases} x_1 + y_1 = 0 \\ y_1 - 2z_1 = 0 \end{cases}, \qquad \begin{cases} x_2 + y_2 = 0 \\ y_2 - 2z_2 = 0 \end{cases}$$
   And test if their sum is also in W:
   $$(x_1 + x_2) + (y_1 + y_2) = x_1 + y_1 + x_2 + y_2 = 0 + 0 = 0$$
   $$(y_1 + y_2) - 2(z_1 + z_2) = y_1 - 2z_1 + y_2 - 2z_2 = 0 + 0 = 0$$
   Now test whether W is closed under scalar multiplication:
   $$kx + ky = k(x + y) = 0$$
   $$ky - 2kz = k(y - 2z) = 0$$
   Therefor, W is indeed a vector space.

   <u>Solution II</u>

   In fact we could see that W is a vector subspace more easily, by noticing that it is in fact a kernel of some linear transformation. We can represent the two defining equations of W in a matrix,
   $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$, such that:

$$W = \{\bar{v} \in \mathbb{R}^3 | A\bar{v} = 0\}$$

Since we have already proved that $\ker(A)$ is a vector subspace, this completes the proof.

3. **Determinants and invertibility**

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & -1 \\ 1 & a^2 & 1 \end{pmatrix}$. For what values of a is A regular?

Solution I

To answer this question, we will calculate $\det(A)$. If $\det(A) \neq 0$, then A is invertible (explain to yourself why). We can perform two row or columns operations:

- If you add a multiple of one row to another, the determinant remains unchanged.
- If you interchange two rows or columns, the sign of the determinant changes:

$$\det(A) \to -\det(A)$$

We can try to get as many zeros in the first column to get an easier calculation:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & -1 \\ 1 & a^2 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & -2 \\ 0 & a^2-1 & 0 \end{pmatrix}$$

So:

$$\det(A) = 0 - (a^2 - 1) \cdot (-2) = 2(a+1)(a-1)$$
$$\det(A) = 0 \leftrightarrow a = \pm 1.$$

Therefor A is regular for $a \neq \pm 1$.

Solution II

A is regular if the columns of A are linearly independent. We can see that the columns are dependent if $a = 1$ (because the first two columns are the same) and also if -1 (because the last two columns are the same). This already gives us two values for which the matrix singular. Should we look for other values as well? No: if we calculate $\det(A)$ using the second column, we can see that we will get some polynomial in a, with the highest power being a^2. Therefore, the two roots we found are the only two roots.

4. Let A be a matrix whose columns are $\bar{v}_1, \bar{v}_2, \bar{v}_3$.

Determine if there is enough information to answer the following questions. If so, give an answer. If not, explain why or give a counter example.

4.1. Solve $A\bar{x} = \bar{v}_1 - \bar{v}_2 + \bar{v}_3$

4.2. Suppose that $\bar{v}_1 - \bar{v}_2 + \bar{v}_3 = 0$. Can $A\bar{x} = \bar{b}$ have a unique solution?

4.3. Unrelated to previous sections: Suppose $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are orthonormal. What linear combination of $\bar{v}_1$ and $\bar{v}_2$ is closest to $\bar{v}_3$?

Solution

5.1 A solution $\bar{x}$ to $A\bar{x} = \bar{b}$ is the set of coefficients that give the correct linear combination of the columns of A. Therefore, $\bar{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

5.2 In this case there must be infinite solutions, because if $\bar{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a solution, then so is $k\bar{x}$ for any scalar $k \in \mathbb{R}$.
Another solution: we know that a solution is unique only if $\ker(A) = \{0\}$ (remember why?). The question tells us that there is a non-zero vector $\bar{x} \in \ker(A)$, and therefore the solution cannot be unique.

5.3 We are actually looking for the projection of $\bar{v}_3$ onto the subspace spanned by $\bar{v}_1, \bar{v}_2$. Since they are all orthonormal, this projection is in fact the zero vector (think about the case of $\hat{\imath}, \hat{\jmath}, \hat{k}$):

$$(\bar{v}_1^T \bar{v}_3)\bar{v}_1 + (\bar{v}_2^T \bar{v}_3)\bar{v}_2 = 0$$

5. True or false?
Let A be an $m \times n$ matrix. $A^T A$ is invertible if and only if the columns of A are linearly independent.

Solution

True. $A^T A$ is invertible if and only if $\ker(A^T A) = \{0\}$. We saw in class that $\ker(A^T A) = \ker(A)$. Finally, if the columns of A are linearly independent, then indeed $\ker(A) = \{0\}$.