Supplementary Math Course -Linear Algebra (76967)

Final Exam (semester A – MOED A) 20.2.2023

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Let $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, b = \begin{bmatrix} 0 \\ a \\ 2 \end{bmatrix}$.
   a. We want to be able to express the vector b as a linear combination of the vectors a_1 and a_2. Write this as a system of equations, in matrix form ($A\bar{x} = \bar{b}$). [7 pt]
   b. Determine all the values of a such that the corresponding system is consistent. [10 pts]
   c. For which value(s) of a the vector b is a linear combination of the vectors a_1 and a_2? [4 pt]
   d. Solve the system of equations for this value a of and write b as a linear combination of the vectors a_1 and a_2. [4 pt]

(25 pts)

2. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.
   a) Find the characteristic polynomial of A. [7 pts]
   b) What are the eigenvalues of A? What is the algebraic multiplicity of each eigenvalue? [4 pts]
   c) Find a basis for the eigenspace of the eigenvalue with the highest algebraic multiplicity. [10 pts]
   d) What is the geometric multiplicity of the eigenvalue with the highest algebraic multiplicity? [4 pts]

(25 pts)

3. Let T: $\mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$
   a) Find the matrix representation of T in the basis $\left\{\bar{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\}$. [3 pts]
   b) Find the matrix representation of T in the standard basis. [10 pts]
   c) What is $rank(T)$? Explain. [7 pts]
   d) What is $ker(T)$? Explain. [5 pts]

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. Let A and B be n×n matrices. If these matrices have a common eigenvector, then det(AB−BA)=0.
2. The set $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}\right\}$ is an orthogonal basis of Span(S), where S=$\left\{\bar{v}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \bar{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$.
3. If A and B are n×n symmetric matrices, then the sum A+B is also symmetric.
4. If v_1, v_2, v_3 are linearly dependent, then v_1, v_2, v_3, v_4 are linearly dependent.
5. If the coefficient matrix of a system of linear equations is singular, then the system is inconsistent.

(5 pts each)

Good luck to all of you!