Supplementary Math Course -Linear Algebra (76967)

Final Exam (semester A – MOED A) 20.2.2023- **solution**

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Let $a_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, a_2 = \begin{bmatrix}2\\-1\\4\end{bmatrix}, b = \begin{bmatrix}0\\a\\2\end{bmatrix}$.

a. We want to be able to express the vector b as a linear combination of the vectors a_1 and a_2. Write this as a system of equations, in a matrix form ($A\bar{x} = \bar{b}$). [7 pt]

To be able to express the vector $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, there must be scalars c_1, c_2 such that

$$c_1\mathbf{a} + c_2\mathbf{a}_2 = \mathbf{b}.$$

This is equivalent to the matrix equation

$$A\begin{bmatrix}c_1\\c_2\end{bmatrix} = \mathbf{b},$$

where

$$A = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix}1 & 2\\2 & -1\\3 & 4\end{bmatrix}.$$

Thus, the vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent.

b. Determine all the values of a such that the corresponding system is consistent. [10 pts]

So let us consider the augmented matrix of the system and reduce it by elementary row operations.

We have

$$[A \mid \mathbf{b}] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & a \\ 3 & 4 & 2 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & a \\ 0 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & a \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -5 & a \end{array}\right]$$

$$\xrightarrow[R_3+5R_2]{R_1-2R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & a-5 \end{array}\right].$$

If $a - 5 = 0$, then we obtain the solution $x_1 = 2, x_2 = -1$. Thus the system is consistent when $a = 5$.

On the other hand, if $a - 5 \neq 0$, then we divide the third row by $a - 5$ and then the third row becomes $\left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$, which implies that the system is inconsistent (as we have $0 = 1$.)

Therefore, the only possible value for a is $a = 5$.

c. For which value(s) of a the vector b is a linear combination of the vectors a_1 and a_2? [4 pt]
The vector b is a linear combination of a_1 and a_2 if and only if the system Ax=b is consistent, which is for a=5.

d. Solve the system of equations for this value a and write b as a linear combination of the vectors a_1 and a_2. [4 pt]
The solution of the system is $x_1 = 2, x_2 = -1$, and therefore, $2a_1 - a_2 = b$.

(25 pts)

2. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

a. Find the characteristic polynomial of A. [7 pts]

By definition, the characteristic polynomial of A is $p(t) = \det(A - tI)$.

We have

$$\begin{aligned}
p(t) &= \det(A - tI) \\
&= \begin{vmatrix} -t & 0 & 0 & 0 \\ 1 & 1-t & 1 & 1 \\ 0 & 0 & -t & 0 \\ 1 & 1 & 1 & 1-t \end{vmatrix} \\
&= -t \begin{vmatrix} 1-t & 1 & 1 \\ 0 & -t & 0 \\ 1 & 1 & 1-t \end{vmatrix} && \text{by the first row cofactor expansion} \\
&= -t \left(-t \begin{vmatrix} 1-t & 1 \\ 1 & 1-t \end{vmatrix} \right) && \text{by the second row cofactor expansion} \\
&= t^2 \left((1-t)^2 - 1 \right) \\
&= t^2(t^2 - 2t) \\
&= t^3(t-2).
\end{aligned}$$

b. What are the eigenvalues of A? What is the algebraic multiplicity of each eigenvalue? [4 pts]

Eigenvalues and their algebraic multiplicities are determined by the characteristic polynomial of A.

From this, the eigenvalues of A are 0 and 2 with algebraic multiplicities 3 and 1, respectively.

c. Find a basis for the eigenspace of the eigenvalue with the highest algebraic multiplicity. [10 pts]

The eigenvalue with the highest algebraic multiplicity is λ=0, and its eigenspace is E0.

By definition $E_0 = \mathcal{N}(A - 0I) = \mathcal{N}(A)$.

Thus, the eigenspace E_0 is the null space of the matrix A.

We solve the equation $A\mathbf{x} = \mathbf{0}$ as follows.

The augmented matrix of this equation is

$$[A \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_4 - R_2} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence the solution satisfies

$$x_1 = -x_2 - x_3 - x_4$$

and the general solution is

$$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the eigenspace is

$$E_0 = \mathcal{N}(A)$$
$$= \left\{ \mathbf{x} \in \mathbb{C}^4 \;\middle|\; \mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ for any } x_2, x_3, x_4 \in \mathbb{C} \right\}$$
$$= \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus the set

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a spanning set of E_0, and it is straightforward to check that the set is linearly independent.

d. What is the geometric multiplicity of the eigenvalue with the highest algebraic multiplicity? [4 pts]

This set is a basis of E0, and the dimension of E0 is 3.
The geometric multiplicity of λ=0 is the dimension of E0 by definition.
Thus, the geometric multiplicity of λ is 3.

(25 pts)

3. Let T: $\mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$

a. Find the matrix representation of T in the basis $\left\{\bar{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\}$. [3 pts]

The columns of the matrix are the images of the corresponding basis vectors:

$$A_B = \begin{bmatrix} 1 & 0 \\ 2 & -5 \\ 3 & 1 \end{bmatrix}$$

b. Find the matrix representation of T in the standard basis. [10 pts]

The matrix representation A of the linear transformation is given by

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)],$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for $\mathbb{R}^2$.

To determine $T(\mathbf{e}_1)$, we first express $\mathbf{e}_1$ as a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as follows.

Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ is $3 \cdot 3 - 4 \cdot 2 = 1 \neq 0$ and thus it is invertible.

Hence we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Hence $a = 3$ and $b = -2$.

It yields that

$$\mathbf{e}_1 = 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

It follows that

$$\begin{aligned} T(\mathbf{e}_1) &= T\left(3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \\ &= 3T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) - 2T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) && \text{by linearity of } T \\ &= 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 7 \end{bmatrix}. \end{aligned}$$

Similarly, we compute $T(\mathbf{e}_2)$ as follows.

Let

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 3 \\ 2 \end{bmatrix} + d \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, we obtain

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix},$$

and $c = -4, d = 3$.

Hence

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

and we have

$$T(\mathbf{e}_2) = T\left(-4\begin{bmatrix}3\\2\end{bmatrix} + 3\begin{bmatrix}4\\3\end{bmatrix}\right)$$

$$= -4T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) + 3T\left(\begin{bmatrix}4\\3\end{bmatrix}\right) \quad \text{by linearity of } T$$

$$= -4\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}0\\-5\\1\end{bmatrix} = \begin{bmatrix}-4\\-23\\-9\end{bmatrix}.$$

Therefore the matrix representation A of T is

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}.$$

c. What is $rank(T)$? Explain. [7 pts]

Let us first determine the rank of A.

We have

$$A = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix}3 & -4\\16 & -23\\1 & -1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_3} \begin{bmatrix}1 & -1\\16 & -23\\3 & -4\end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-16R_1} \begin{bmatrix}1 & -1\\0 & -7\\0 & -1\end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix}1 & -1\\0 & -7\\0 & 1\end{bmatrix} \xrightarrow[R_2+7R_3]{R_1+R_3} \begin{bmatrix}1 & 0\\0 & 0\\0 & 1\end{bmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}.$$

Hence, the reduced row echelon form matrix of A has two nonzero rows.

So the rank of A is 2.

d. What is $ker(T)$? Explain. [5 pts]

By the rank-nullity theorem, for a linear transformation $T: V \to U$, we have:

$$rank(T) + \underbrace{nullity(T)}_{\dim(ker(T))} = \dim(V)$$

In this case $T: \mathbb{R}^2 \to \mathbb{R}^3$, so:

$$2 + \dim(\ker(A)) = 2 \quad \to \quad \dim(\ker(A)) = 0$$

This means that the only vector in the null-space of T is the 0 vector: $\ker(T) = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$.

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. Let A and B be n×n matrices. If these matrices have a common eigenvector, then det(AB−BA)=0.

True. Let α and β be eigenvalues of A and B such that the vector x is a corresponding eigenvector. Namely we have Ax=αx and Bx=βx. Then we have:
(AB−BA)x=ABx−BAx=A(βx)−B(αx)=βAx−αBx=βα−αβ=0.
By the definition of eigenvector, x is a non-zero vector, which means that ker(AB−BA) is not 0. Thus, the matrix AB−BA is singular (non-invertible). Equivalently the determinant of AB−BA is zero.

2. The set $\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}\}$ is an orthogonal basis of Span(S), where S=$\{\bar{v}_1=\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\bar{v}_2=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\}$.

True.

It is straightforward to check that the vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, and hence the set S is a basis of Span(S).
Since the dot (inner) product of $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\mathbf{v_1} \cdot \mathbf{v}_2 = 1 \neq 0,$$

S is not an orthogonal basis. We apply the Gram-Schmidt process to generate an orthogonal basis from the basis S.

The Gram-Schmidt process for two vectors is as follows. We define vectors $\mathbf{u}_1, \mathbf{u}_2$ by the following formula. Then $B = \{\mathbf{u}_1, \mathbf{u_2}\}$ is an orthogonal basis of Span(S).

$$\mathbf{u}_1 := \mathbf{v}_1$$

$$\mathbf{u}_2 := \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1. \qquad (*)$$

Since we have

$$\mathbf{u}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \text{ and } \mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{v}_1 \cdot \mathbf{v}_1 = 2,$$

we compute

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{1}{2}\mathbf{u}_1$$
$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis of Span(S).
Note that scaling by a nonzero scalar does not change the orthogonality, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is also an orthogonal basis of Span(S), just in case you prefer not to have a fraction.

3. If A and B are n×n symmetric matrices, then the sum A+B is also symmetric.

True. Since A and B are symmetric, we have $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$.
It follows that

$$(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = A + B.$$

Thus the sum $A + B$ is symmetric.

4. If v_1, v_2, v_3 are linearly dependent, then v_1, v_2, v_3, v_4 are linearly dependent.

True. Since the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, there exists scalars c_1, c_2, c_3, not all of them are zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Let $\mathbf{v}_4$ be any n-dimensional vector.
Then we have the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$$

whose coefficient is not trivial as at least one of c_1, c_2, c_3 is nonzero.
This implies that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is also linearly dependent.

5. If the coefficient matrix of a system of linear equations is singular, then the system is inconsistent.

False. The system could be consistent even though the coefficient matrix is singular.
For example, consider the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

The coefficient matrix of the system is $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
This is singular since, for example, it is row equivalent to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
However, the system has a solution, for example, $x_1 = 1, x_2 = 1$.
Hence the system is consistent even though its coefficient matrix is singular.

(5 pts each)

Good luck to all of you!