Supplementary Math Course -Linear Algebra (76967)

Final Exam (summer) 28.10.2022

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations, with parameter $a \in \mathbb{R}$:

$$\begin{aligned} x + 2y + z &= 0 \\ -x - y + z &= 0 \\ 3x + 4y + az &= 0 \end{aligned}$$

   a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]
   b. Assume there are values of a so that the system has nontrivial solution. For those values, determine how many solutions there are to the system **without finding the solutions of the system**. Explain. [5 pt]
   c. Determine all the values of a so that the system has nontrivial solution. [10 pts]
   d. For the values of a you found such that the system has nontrivial solution, find the solutions of the system. [5 pt]
   e. For what values of a Is A invertible? Explain [4 pt]

(25 pts)

2. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$. The matrix has an eigenvalue 2.

   a. Find a basis of the eigenspace corresponding to the eigenvalue 2 (which is a basis for the eigenvectors corresponding to the eigenvalue 2). [10 pts]
   b. Find the other eigenvalues of A. [5 pts]
   c. Find the eigenvectors corresponding to the eigenvalues you found in section b. [5 pts]
   d. Find the determinant of A. [5 pts]

(25 pts)

3. Let $B = \left\{ \bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \bar{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ be a basis to $\mathbb{R}^2$, and let T be a linear transformation defined by:

$$T(\bar{u}) = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, T(\bar{v}) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

   a. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis). [3 pts]
   b. Let $\bar{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. Find the formula for $T(\bar{w})$ in terms of x and y. (Hint: Write $\bar{w}$ as a linear combination of $\bar{u}$ and $\bar{v}$.) [12 pts]
   c. Find the matrix representation of T in the standard basis. [5 pts]
   d. Find a basis for $Im(T)$. [5 pts]

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. Let A and B be n×n matrices (n>1). Then: $\det(A + B) = \det(A) + \det(B)$.

2. Let $A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7+x & -3 \\ 0 & 4 & x \end{bmatrix}$. The matrix A is invertible for all x except x=−3 and x=−4.

3. The matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is diagonalizable.

4. Every diagonalizable matrix is invertible.

5. Every invertible matrix is diagonalizable.

(5 pts each)

Good luck to all of you!