Supplementary Math Course -Linear Algebra (76967)

Final Exam (summer) 28.10.2022- **solution**

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations, with parameter $a \in \mathbb{R}$:

$$x + 2y + z = 0$$
$$-x - y + z = 0$$
$$3x + 4y + az = 0$$

a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]

The matrix form of the system ($A\bar{x} = \bar{b}$):

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 3 & 4 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b. Assume there are values of a so that the system has nontrivial solution. For those values, determine how many solutions there are to the system **without finding the solutions of the system**. Explain. [5 pt]

The system is homogeneous and hence it is consistent. Thus, if the system has a nontrivial solution, then it has infinitely many solutions.

c. Determine all the values of a so that the system has nontrivial solution. [10 pts]

The system has a nontrivial solution, when it has infinitely many solutions.

This happens if and only if the system has at least one free variable. The number of free variables is n−r, where n is the number of unknowns and r is the rank of the augmented matrix.

To find the rank, we reduce the augmented matrix by elementary row operations.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 3 & 4 & a & 0 \end{array}\right] R_2 + R_1 \,\&\, R_3 - 3R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & a-3 & 0 \end{array}\right]$$

$$R_3 + 2R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & a+1 & 0 \end{array}\right]$$

If $a + 1 = 0$, then the third row is a zero row, hence the rank is 2. In this case we have n−r=3−2=1 free variable. Thus there are infinitely many solutions. In particular, the system has nontrivial solutions.

On the other hand, if $a + 1 \neq 0$, then the rank is 3 and there is no free variables since n−r=3−3=0.

In summary, the system has nontrivial solutions exactly when $a = -1$.

d. For the values of a you found such that the system has nontrivial solution, find the solutions of the system. [5 pt]

$a = -1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can replace the free-variable z with a parameter c to remind ourselves that it can be assigned any value. The general parametric solution then becomes:

$$(x, y, z) = (3\text{c}, -2\text{c}, \text{c})$$

e. For what values of a Is A invertible? Explain [4 pt]

For $a \neq -1$.

(25 pts)

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2. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$. The matrix has an eigenvalue 2.

a. Find a basis of the eigenspace corresponding to the eigenvalue 2 (which is a basis for the eigenvectors corresponding to the eigenvalue 2). [10 pts]

By definition, the eigenspace E_2 corresponding to the eigenvalue 2 is the null space of the matrix $A - 2I$.

That is, we have

$$E_2 = \mathcal{N}(A - 2I).$$

We reduce the matrix $A - 2I$ by elementary row operations as follows.

$$A - 2I = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3+2R_1]{R_2-R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the solutions $\mathbf{x}$ of $(A - 2I)\mathbf{x} = \mathbf{0}$ satisfy $x_1 = 2x_2 + x_3$.

Thus, the null space $\mathcal{N}(A - 2I)$ consists of vectors

$$\mathbf{x} = \begin{bmatrix} 2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for any scalars x_2, x_3.

Hence we have

$$E_2 = \mathcal{N}(A - 2I) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

It is straightforward to see that the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, hence they form a basis of E_2.

Thus, a basis of E_2 is

$$\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

b. Find the other eigenvalues of A. [5 pts]

$$\det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

We know that $\lambda = 2$ is a solution, so we can write the Characteristic polynomial as $(\lambda - 2)(\ldots?)$→

$$\det(A - \lambda I) = (\lambda - 2)(-\lambda^2 + 3\lambda - 2) = (\lambda - 2)^2(1 - \lambda) = 0$$

So, the other eigenvalue is $\lambda = 1$.

c. Find the eigenvectors corresponding to the eigenvalues you found in section b. [5 pts]

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is the solution of $(A - I)x = 0$:

$$(A - I) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 2 & -4 & -1 \end{bmatrix} \; R_1 \to R_1 - R_2 \;\&\; R_2 \to R_2 + \tfrac{1}{2}R_3 \;\&\; R_3 \to R_3 + 2R_2 :$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & 1 \end{bmatrix} \; R_3 \to -2R_2 : \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We can replace the free-variable z with a parameter c. The general parametric solution then becomes:

$$(x, y, z) = c\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

d. Find the determinant of A [5 pts]

The product of the eigenvalues equals the determinant: $\det(A) = 2 * 2 * 1 = 4$. Notice that we could do that because the geometric multiplicity of eigenvalue 2 is equal to the algebraic multiplicity of eigenvalue 2 (we got power of 2 to the term $(\lambda - 2)$ in the

Characteristic polynomial and the dimension of the eigenspace corresponding to the eigenvalue 2 is 2).

3. Let $B = \left\{ \bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \bar{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ be a basis to $\mathbb{R}^2$, and let T be a linear transformation defined by:

$$T(\bar{u}) = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, T(\bar{v}) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

a. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis). [3 pts]

The columns of the matrix are the images of the corresponding basis vectors:

$$A_B = \begin{bmatrix} -3 & 7 \\ 5 & 1 \end{bmatrix}$$

b. Let $\bar{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. Find the formula for $T(\bar{w})$ in terms of x and y. (**Hint**: Write $\bar{w}$ as a linear combination of $\bar{u}$ and $\bar{v}$.) [12 pts]

Note that the vectors $\mathbf{u}$, $\mathbf{v}$ are basis vectors for $\mathbb{R}^2$.

Thus we can write the vector $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Let

$$a_1\mathbf{u} + a_2\mathbf{v} = \mathbf{w}.$$

We want to determine a_1 and a_2.

So we consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 2 & 5 & y \end{array}\right].$$

Applying the elementary row operations, we obtain a reduced row echelon form matrix for this matrix as follows. (This is the Gauss-Jordan elimination method.)

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 2 & 5 & y \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -1 & y-2x \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & 2x-y \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & x-3(2x-y) \\ 0 & 1 & 2x-y \end{array}\right].$$

Therefore we have

$$a_1 = x - 3(2x-y) = -5x + 3y$$
$$a_2 = 2x - y$$

and the linear combination is

$$\mathbf{w} = (-5x+3y)\mathbf{u} + (2x-y)\mathbf{v}.$$

Now we use the linearity of the linear transformation T, we calculate $T(\mathbf{w})$ as follows.

$$\begin{aligned} T(\mathbf{w}) &= T((-5x+3y)\mathbf{u} + (2x-y)\mathbf{v}) \\ &= (-5x+3y)T(\mathbf{u}) + (2x-y)T(\mathbf{v}) \\ &= (-5x+3y)\begin{bmatrix} -3 \\ 5 \end{bmatrix} + (2x-y)\begin{bmatrix} 7 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 15x-9y+14x-7y \\ -25x+15y+2x-y \end{bmatrix} \\ &= \begin{bmatrix} 29x-16y \\ -23x+14y \end{bmatrix}. \end{aligned}$$

Thus the formula is

$$T(\mathbf{w}) = \begin{bmatrix} 29x-16y \\ -23x+14y \end{bmatrix}.$$

c. Find the matrix representation of T in the standard basis. [5 pts]

We can use the formula we found:

When $\overline{w} = \hat{e}_1 \rightarrow = T(\overline{w}) = \begin{bmatrix} 29 \\ -23 \end{bmatrix}$

When $\overline{w} = \hat{e}_2 \rightarrow = T(\overline{w}) = \begin{bmatrix} -16 \\ 14 \end{bmatrix}$

Therefore:

$$A_S = \begin{bmatrix} 29 & -16 \\ -23 & 14 \end{bmatrix}$$

d. Find a basis for $Im(T)$. [5 pts]

The image of T is spanned by the columns of A .The columns of A_S are linearly independent (show that!) and therefore a basis for $Im(T)$ is:

$$\left\{\begin{bmatrix} 29 \\ -23 \end{bmatrix}, \begin{bmatrix} -16 \\ 14 \end{bmatrix}\right\}$$

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. Let A and B be n×n matrices (n>1). Then: $\det(A+B) = \det(A) + \det(B)$.

False.

We claim that the statement is false.

As a counterexample, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we have

$$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

On the other hand, the determinants of A and B are

$$\det(A) = 0 \text{ and } \det(B) = 0,$$

and hence

$$\det(A) + \det(B) = 0 \neq 1 = \det(A+B).$$

Therefore, the statement is false and in general we have

$$\det(A+B) \neq \det(A) + \det(B).$$

2. Let $A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7+x & -3 \\ 0 & 4 & x \end{bmatrix}$. The matrix A is invertible for all x except x=−3 and x=−4.

True.

A matrix is invertible if and only if its determinant is non-zero.

So we first calculate the determinant of the matrix A.

By the first column cofactor expansion, we have

$$\begin{aligned} \det(A) &= 2\begin{vmatrix} 7+x & -3 \\ 4 & x \end{vmatrix} \\ &= 2\left((7+x)x - (-3)4\right) = 2(x^2 + 7x + 12) \\ &= 2(x+3)(x+4). \end{aligned}$$

Thus the determinant of A is zero if and only if $x = -3$ or $x = -4$.

Therefore the matrix A is invertible for all x except $x = -3$ and $x = -4$.

3. The matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is diagonalizable.

True.

To determine whether the matrix A is diagonalizable, we first find eigenvalues of A.

To do so, we compute the characteristic polynomial $p(t)$ of A:

$$p(t) = \begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = (1-t)(3-t) - 8$$
$$= t^2 - 4t - 5 = (t+1)(t-5).$$

The roots of the characteristic polynomial $p(t)$ are eigenvalues of A.

Hence the eigenvalues of A are -1 and 5.

Since the 2×2 matrix A has two distinct eigenvalues, it is diagonalizable.

4. Every diagonalizable matrix is invertible.

False, we give a counterexample: Consider the 2×2 zero matrix. The zero matrix is a diagonal matrix, and thus it is diagonalizable. However, the zero matrix is not invertible as its determinant is zero.

5. Every invertible matrix is diagonalizable.

False.

Note that it is not true that every invertible matrix is diagonalizable.

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The determinant of A is 1, hence A is invertible.

The characteristic polynomial of A is

$$p(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 \\ 0 & 1-t \end{vmatrix} = (1-t)^2.$$

Thus, the eigenvalue of A is 1 with algebraic multiplicity 2.

We have

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and thus eigenvectors corresponding to the eigenvalue 1 are

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for any nonzero scalar a.

Thus, the geometric multiplicity of the eigenvalue 1 is 1.

Since the geometric multiplicity is strictly less than the algebraic multiplicity, the matrix A is defective and not diagonalizable.

(5 pts each)

Good luck to all of you!