Supplementary Math Course -Linear Algebra (76967)

Final Exam - MOED A (semester A) 7.2.2022

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations, with parameter $a \in \mathbb{R}$:

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x - y - 2z &= a^2 \\ -x - 7y - 11z &= a \end{aligned}$$

   a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]
   b. Determine all the values of a so that the corresponding system is consistent. [10 pts]
   c. Find det(A). Is A invertible? [4 pt]
   d. For all the values of a you found such that the system is consistent, determine how many solutions there are to the system **without finding the solutions of the system**. Explain. [5 pt]
   e. For all the values of a you found such that the system is consistent, find the solutions of the system. [5 pt]

(25 pts)

2. Let $A = \begin{bmatrix} \frac{3}{2} & 2 \\ -1 & -\frac{3}{2} \end{bmatrix}$.

   a) What are the eigenvalues of A? [5 pts]

   b) Find a regular matrix P and a diagonal matrix D such that $D = P^{-1}AP$. [10 pts] (Notice that you do not have to find P^{-1})

   c) Let $\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. find $A^{\mathrm{n}}\bar{v}$. Simplify your answer as much as possible. [7 pts]

   d) Show that we can choose n large enough so that the length $|A^{\mathrm{n}}\bar{v}|$ is as small as we want. [3 pts]

(25 pts)

3. Let $B = \left\{ \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis to $\mathbb{R}^2$, and let T be a linear transformation defined by:

$$T(\bar{v}_1) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, T(\bar{v}_2) = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$$

   a. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis). [3 pts]

   b. Find the matrix representation of T in the standard basis. [7 pts]

   c. Find a basis for $Im(T)$. [5 pts]

   d. What is $\mathrm{r}ank(T)$? Explain. [5 pts]

   e. What is $\mathrm{k}er(T)$? Explain. [5 pts]

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. The vectors $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix} \}$ are linearly independent for any value of h.

2. If the system Ax=b has a unique solution, then A must be a square matrix.

3. A linear system with fewer equations than unknowns must have infinitely many solutions.

4. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix such that $a_{11} + a_{12} = 1 \; and \; a_{21} + a_{22} = 1$. Namely, the sum of the entries in each row is 1. The matrix A has an eigenvalue 1.

5. The matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible and the inverse matrix is: $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(5 pts each)

Good luck to all of you!