Supplementary Math Course -Linear Algebra (76967)

Final Exam (winter) 7.2.2022- **solution**

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations, with parameter $a \in \mathbb{R}$:

$$x + 2y + 3z = 4$$
$$2x - y - 2z = a^2$$
$$-x - 7y - 11z = a$$

a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]

The matrix form of the system ($A\bar{x} = \bar{b}$) :

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ -1 & -7 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ a^2 \\ a \end{pmatrix}$$

b. Determine all the values of a so that the corresponding system is consistent. [10 pts]

A system of equations is called consistent if there is at least one set of values for the unknowns that satisfies each equation in the system—that is,
when substituted into each of the equations, they make each equation hold true as an identity.
We apply the elementary row operations to A as follows.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & a^2 \\ -1 & -7 & -11 & a \end{array}\right] R_2 - 2R_1 \;\&\; R_3 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -8 & a^2 - 8 \\ 0 & -5 & -8 & a + 4 \end{array}\right]$$
$$R_3 - R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -8 & a^2 - 8 \\ 0 & 0 & 0 & -a^2 + a + 12 \end{array}\right]$$

The system is consistent if and only if $-a^2 + a + 12$ is zero.
This is because if $-a^2 + a + 12 \neq 0$, then we divide the third row by $-a^2 + a + 12$ and obtain [0 0 0 | 1] in the third row. The corresponding linear equation is 0x+0y+0z=1, and clearly, there is no solution to this equation. Hence the system is inconsistent.
On the other hand, if $-a^2 + a + 12 = 0$, then the system has solutions.

So, we want to find the values of a such that $-a^2 + a + 12 = 0$:

$$-a^2 + a + 12 = -(a + 3)(a - 4) = 0$$

we see that the system is consistent if and only if a =−3,4.

Another way:

You can find a basis for $Im(A)$,:

To find a basis of $Im(A)$, we can row-reduce A^T (this will allow us to find a basis for the rows of A^T, which correspond to the columns of A):

$$\begin{pmatrix}1 & 2 & -1\\ 2 & -1 & -7\\ 3 & -2 & -11\end{pmatrix} R_3 - R_1 - R_2 \rightarrow \begin{pmatrix}1 & 2 & -1\\ 2 & -1 & -7\\ 0 & -3 & -3\end{pmatrix} R_2 - 2R_1 \rightarrow \begin{pmatrix}1 & 2 & -1\\ 0 & -5 & -5\\ 0 & -3 & -3\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}1 & 2 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

Using the two pivots, a basis for $Im(A)$ is:

$$\left\{\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}\right\}$$

Now, for the system to be consistent, find the values of a such that $\bar{b} \in Im(A)$.

(Indeed: For a =-3: $\bar{b} = \begin{pmatrix}4\\ 9\\ -3\end{pmatrix} = 4\begin{pmatrix}1\\ 2\\ -1\end{pmatrix} + \begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$. And for a =4: $\bar{b} = \begin{pmatrix}4\\ 16\\ 4\end{pmatrix} = 4\begin{pmatrix}1\\ 2\\ -1\end{pmatrix} + 8\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$.)

c. Find det(A). Is A invertible? [4 pt]

det(A)=0. A is not invertible.

d. For all the values of a you found such that the system is consistent, determine how many solutions there are to the system **without finding the solutions of the system.** Explain. [5 pt]

When det(A) = 0-> the matrix A is not invertible-> The system of equations either has no solution at all, or it has infinite solutions. However, we consider the case that the system is consistent, so it has infinite solutions.

e. For all the values of a you found such that the system is consistent, find the solutions of the system. [5 pt]

$\underline{a = -3:}$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\ 0 & -5 & -8 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$$

We can replace the free-variable z with a parameter c to remind ourselves that it can be assigned any value. The general parametric solution then becomes:

$$(x, y, z) = (\frac{c + 22}{5}, \frac{-1 - 8c}{5}, \text{c})$$

$\underline{a = 4:}$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\ 0 & -5 & -8 & 8\\ 0 & 0 & 0 & 0\end{array}\right]$$

We can replace the free-variable z with a parameter c to remind ourselves that it can be assigned any value. The general parametric solution then becomes:

$$(x, y, z) = (\frac{c + 36}{5}, \frac{-8 - 8c}{5}, \text{c})$$

(25 pts)

2. Let $A = \begin{bmatrix} \frac{3}{2} & 2 \\ -1 & -\frac{3}{2} \end{bmatrix}$.

a. What are the eigenvalues of A? [5 pts]

To find the eigenvalues of A, we compute the characteristic polynomial $p(t)$ of A.
We have

$$\begin{aligned} p(t) &= \det(A - tI) \\ &= \begin{vmatrix} 3/2 - t & 2 \\ -1 & -3/2 - t \end{vmatrix} \\ &= t^2 - 1/4. \end{aligned}$$

Since the eigenvalues are roots of the characteristic polynomials, the eigenvalues of A are $\pm 1/2$.

b. Find a regular matrix P and a diagonal matrix D such that $D = P^{-1}AP$. [10 pts] (Notice that you do not have to find P^{-1})

Next we find the eigenvectors corresponding to eigenvalue $1/2$.
These are the solutions of $(A - \frac{1}{2}I)\mathbf{x} = \mathbf{0}$.
We have

$$A - \frac{1}{2}I = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Thus, the solution $\mathbf{x}$ satisfies $x_1 = -2x_2$, and the eigenvectors are

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

where x_2 is a nonzero scalar.

Similarly, we find the eigenvectors corresponding to the eigenvalue $-1/2$.
We solve $(A + \frac{1}{2}I)\mathbf{x} = \mathbf{0}$.
We have

$$A + \frac{1}{2}I = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \xrightarrow[\text{then } R_2+R_1]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, we have $x_1 = -x_2$, and the eigenvectors are

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where x_2 is a nonzero scalar.

Therefore:

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad P = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

c. Let $\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. find $A^n \bar{v}$. Simplify your answer as much as possible. [7 pts]

We express the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $1/2$ and $-1/2$, respectively.
Let

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for some scalars c_1, c_2.
Solving this for c_1, c_2, we find that $c_1 = -1$ and $c_2 = 1$, and thus we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then for any positive integer n, we have

$$\begin{aligned} A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -A^n \begin{bmatrix} -2 \\ 1 \end{bmatrix} + A^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= -\left(\frac{1}{2}\right)^n \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \left(\frac{1}{2}\right)^n \begin{bmatrix} 2-(-1)^n \\ -1+(-1)^n \end{bmatrix} \end{aligned}$$

Note that in the second equality we used the following fact: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^n\mathbf{x} = \lambda^n\mathbf{x}$.

d. Show that we can choose n large enough so that the length $|A^n \bar{v}|$ is as small as we want. [3 pts]

Then the length is

$$\left\| A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \left(\frac{1}{2} \right)^n \sqrt{\left(2 - (-1)^n \right)^2 + \left(-1 + (-1)^n \right)^2}$$
$$\leq \left(\frac{1}{2} \right)^n \sqrt{3^2 + 2^2}$$
$$= \sqrt{13} \left(\frac{1}{2} \right)^n \to 0 \text{ as } n \text{ tends to infinity.}$$

Therefore, we can choose n large enough so that the length $\|A^n \mathbf{v}\|$ is as small as we wish.

(25 pts)

3. Let $B = \left\{ \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis to $\mathbb{R}^2$, and let T be a linear transformation defined by:

$$T(\bar{v}_1) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, T(\bar{v}_2) = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$$

a. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis). [3 pts]

The columns of the matrix are the images of the corresponding basis vectors:

$$A_B = \begin{bmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

b. Find the matrix representation of T in the standard basis. [7 pts]

Using the basis vectors of B, we can easily find the image of the basis vectors in the standard basis:

$$A_S \hat{e}_1 = A_B \left(\frac{1}{2} \bar{v}_1 + \frac{1}{2} \bar{v}_2 \right) = \begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix}$$
$$A_S \hat{e}_2 = A_B \left(\frac{1}{2} \bar{v}_1 - \frac{1}{2} \bar{v}_2 \right) = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Therefore:

$$A_S = \begin{bmatrix} 1 & 1 \\ 6 & -2 \\ 8 & -2 \end{bmatrix}$$

c. Find a basis for $Im(T)$. [5 pts]

The image of T is spanned by the columns of A .The columns of A_S are linearly independent (show that!) and therefore a basis for $Im(T)$ is:

$$\left\{ \begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right\}$$

d. What is $rank(T)$? Explain. [5 pts]

The rank of T is the rank of the matrix A that represent it. Since A is full rank: $rank(T) = rank(A) = 2$.
(Note that the maximal possible rank is $\min(2,3) = 2$).

e. What is $ker(T)$? Explain. [5 pts]
By the rank-nullity theorem, for a linear transformation $T: V \to U$, we have:

$$rank(T) + \underbrace{nullity(T)}_{\dim(ker(T))} = \dim(V)$$

In this case $T: \mathbb{R}^2 \to \mathbb{R}^3$, so:

$$2 + \dim(\ker(A)) = 2 \quad \to \quad \dim(\ker(A)) = 0$$

This means that the only vector in the null-space of T is the 0 vector: $\ker(T) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$.

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. The vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix} \right\}$ are linearly independent for any value of h.

False. They are linearly independent for any values of h **except** $-1/2$, -1. We give two solutions. The first one uses the homogeneous system and the second one uses a determinant:

a. Solution 1:

Let us consider the linear combination

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}.$$

If this homogeneous system has only zero solution $x_1 = x_2 = x_3 = 0$, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We reduce the augmented matrix for the system as follows.

$$\left[\begin{array}{ccc|c} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & -h & 3h+1 & 0 \end{array}\right] \xrightarrow{R_3+hR_2} \left[\begin{array}{ccc|c} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & 0 & 2h^2+3h+1 & 0 \end{array}\right].$$

From this, we see that the homogeneous system (*) has only the zero solution if and only if

$$2h^2 + 3h + 1 \neq 0.$$

Since we have

$$2h^2 + 3h + 1 = (2h+1)(h+1),$$

if $h \neq -\frac{1}{2}, -1$, then $2h^2 + 3h + 1 \neq 0$.

In summary, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent for any h except $-\frac{1}{2}, -1$.

b. Solution 2:

Note that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only the matrix

$$A := [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \begin{bmatrix} 1 & h & 1 \\ 0 & 1 & 2h \\ 0 & -h & 3h+1 \end{bmatrix}$$

is nonsingular.

Also, the matrix A is nonsingular if and only if the determinant $\det(A)$ is nonzero.

So we compute the determinant of the matrix A by the first column cofactor expansion and obtain

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & h & 1 \\ 0 & 1 & 2h \\ 0 & -h & 3h+1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2h \\ -h & 3h+1 \end{vmatrix} \\ &= 2h^2 + 3h + 1 \\ &= (2h+1)(h+1). \end{aligned}$$

Hence $\det(A) \neq 0$ if and only if $h \neq -\frac{1}{2}, -1$.

Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent for any values of h except $-\frac{1}{2}, -1$.

2. If the system Ax=b has a unique solution, then A must be a square matrix.

False. For example, consider the matrix $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then the system $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has a unique solution x=0 but A is not a square matrix.

3. A linear system with fewer equations than unknowns must have infinitely many solutions.

False. For example, consider the system of one equation with two unknowns 0x+0y=1. This system has no solution at all.

4. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix such that $a_{11} + a_{12} = 1 \; and \; a_{21} + a_{22} = 1$. Namely, the sum of the entries in each row is 1. The matrix A has an eigenvalue 1.

True. Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x.$

5. The matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible and the inverse matrix is: $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

True. Consider the augmented matrix [A|I] with the 3×3 identity matrix I.

Reduce [A|I] using elementary row operations as follows.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

This is in reduced row echelon form and the left 3×3 part is the identity matrix. Hence A is invertible, and the inverse matrix is:

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts each)

Good luck to all of you!