Supplementary Math Course -Linear Algebra (76967)

Final Exam (summer) 13.10.2021

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

   a. What are the eigenvalues of A? [5 pts]

   b. Find a regular matrix P and a diagonal matrix D such that $D = P^{-1}AP$. [10 pts] (Notice that you do not have to find P^{-1})

   c. Let $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be some vector (in the standard basis).

      Let $\lambda_1 > \lambda_2$ be the two eigenvalues of A, with eigenvectors $\bar{u}_1, \bar{u}_2$ (the first eigenvectors of A corresponds to the bigger eigenvalue).
      When represented in the eigenbasis, $\bar{v}$ has the form $\bar{v}_{eb} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
      Find the representation of $\bar{v}$ in the standard basis. [5 pts]

   d. For the same $\bar{v}$, find $A^5\bar{v}$. Simplify your answer as much as possible. [5 pts]

(25 pts)

2. Consider the following system of equations, with parameters $a, b \in \mathbb{R}$:

$$\begin{aligned} &x + 2y + z = 3 \\ &ay + 5z = 10 \\ &2x + 7y + az = b \end{aligned}$$

   a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]

   b. In this section, assume that the system is consistent.
      Which of the parameters affects the existence of a unique solution – a, b or both? Explain. [5 pts]

   c. Find the parameter value(s) for which the system has a unique solution. [9 pts]

   d. Find the parameter value(s) for which the system has more than one solution. [10 pts]

(25 pts)

3. Let $B = \left\{ \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ be a basis to $\mathbb{R}^3$, and let T be a linear transformation defined by:

$$T(\bar{v}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, T(\bar{v}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(\bar{v}_3) = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

a. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis). [3 pts]

b. Find the matrix representation of T in the standard basis. [7 pts]

c. Find a basis for $Im(T)$. [10 pts]

d. Find the dimension of $\ker(T)$. [3 pts]

e. Is T invertible? [2 pts]

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
If it is true, explain or prove shortly. If it is false, give a counter example.
(Answers without a proper explanation will not get any points)

1. Let $A \in M_{n\times n}(\mathbb{R})$. If A is diagonalizable, then so is A^2.

2. Every diagonalizable matrix $A \in M_{n\times n}(\mathbb{R})$ is also invertible.

3. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

4. If A is a real $n \times n$ matrix with linearly independent columns, then there is always a solution to $A\bar{x} = \bar{b}$.

5. If $A^2 = 0$ then also $A = 0$.

(5 pts each)

Good luck to all of you!