Supplementary Math Course -Linear Algebra (76967)

Final Exam (summer) 13.10.2021- **solution**

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

2.1. What are the eigenvalues of A? [5 pts]

$$\det(A - \lambda I) = 0$$

$$det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = 3 - 4\lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5$$

$$= (\lambda - 5)(\lambda + 1) = 0$$

So:

$$\lambda_1 = 5, \lambda_2 = -1$$

2.2. Find a regular matrix P and a diagonal matrix D such that $D = P^{-1}AP$. [10 pts]
(Notice that you do not have to find P^{-1})

We will find eigenvectors corresponding to each of the eigenvalues:

$\underline{\lambda = 5}$

$$(A - \lambda I)\bar{v} = 0$$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 = 2v_2 \to v_2 = 2v_1$$

So we can take:

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\underline{\lambda = -1}$

$$(A - \lambda I)\bar{v} = 0$$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = -v_2$$

So we can take:

$$\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

2.3. Let $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be some vector (in the standard basis).
Let $\lambda_1 > \lambda_2$ be the two eigenvalues of A, with eigenvectors $\bar{u}_1, \bar{u}_2$ (the first eigenvectors of A corresponds to the bigger eigenvalue).
When represented in the eigenbasis, $\bar{v}$ has the form $\bar{v}_{eb} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Find the representation of $\bar{v}$ in the standard basis. [5 pts]

Since $\bar{u}_1 = \binom{1}{2}$ and $\bar{u}_2 = \binom{1}{-1}$, we get that:

$$\bar{v} = 2\binom{1}{2} + \binom{1}{-1} = \binom{3}{3}$$

2.4. For the same $\bar{v}$, find $A^5\bar{v}$. Simplify your answer as much as possible. [5 pts]

In the eigenbasis, for any eigenvector $\bar{u}_i$ we have $A^k\bar{u}_i = \lambda_i^k\bar{u}_i$, so:

$$A^5\bar{v} = A^5(2\bar{u}_1 + \bar{u}_2) = 2\lambda_1^5\bar{u}_1 + \lambda_2^5\bar{u}_2 = 2\cdot 5^5\binom{1}{2} + (-1)^5\binom{1}{-1} = 5^5\binom{2}{4} + \binom{-1}{1}$$

(25 pts)

2. Consider the following system of equations, with parameters $a, b \in \mathbb{R}$:

$$x + 2y + z = 3$$
$$ay + 5z = 10$$
$$2x + 7y + az = b$$

2.1. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]

The matrix form of the system ($A\bar{x} = \bar{b}$) :

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & a & 5 \\ 2 & 7 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ b \end{pmatrix}$$

2.2. In this section, assume that the system is consistent.

Which of the parameters affects the existence of a unique solution – a, b or both? Explain. [5 pts]

Only a affects the existence of a unique solution to the system. A unique solution exists for any vector $\bar{b}$ if $Im(A) = codomain$, and since $Im(A)$ only depends on the columns of the matrix, only the parameter A affects it.

Equivalently, a unique solution exists only when $\det(A) \neq 0$. Since $\det(A)$ only depends on the matrix terms, b has no effect on the existence of a unique solution.

2.3. Find the parameter value(s) for which the system has a unique solution. [9 pts]

A unique solution exist only if $\det(A) \neq 0$. We can calculate $\det(A)$ using the first column:

$$\det(A) = a^2 - 35 + 2(10 - a) = a^2 - 2a - 15 = (a + 3)(a - 5)$$

So for $a \neq -3, +5$ we there is a unique solution.

2.4. Find the parameter value(s) for which the system has more than one solution. [10 pts]

If the system has more than one solution, it has infinite solutions. This is possible when $\det(A) = 0$ (there is linear dependence between the columns) and $\bar{b} \in Im(A)$.

For each possible value of a, We will use Gaussian elimination on the augmented matrix to find the relevant values of b.

We are looking for values of b for which the system is consistent (i.e., there is no contradicting equations):

$\underline{a=-3}$

$$\begin{pmatrix}1 & 2 & 1 & 3\\ 0 & \mathbf{-3} & 5 & 10\\ 2 & 7 & \mathbf{-3} & b\end{pmatrix} R_3-2R_1 \to \begin{pmatrix}1 & 2 & 1 & 3\\ 0 & -3 & 5 & 10\\ 0 & 3 & -5 & b-6\end{pmatrix} R_3+R_2 \to \begin{pmatrix}1 & 2 & 1 & 3\\ 0 & -3 & 5 & 10\\ 0 & 0 & 0 & b+4\end{pmatrix}$$

So the system has infinite solution when:

$$\boldsymbol{a=-3, b=-4}$$

$\underline{a=5}$

$$\begin{pmatrix}1 & 2 & 1 & 3\\ 0 & \mathbf{5} & 5 & 10\\ 2 & 7 & \mathbf{5} & b\end{pmatrix} R_3-2R_1 \to \begin{pmatrix}1 & 2 & 1 & 3\\ 0 & 5 & 5 & 10\\ 0 & 3 & 3 & b-6\end{pmatrix} R_3-\frac{3}{5}R_2 \to \begin{pmatrix}1 & 2 & 1 & 3\\ 0 & 5 & 5 & 10\\ 0 & 0 & 0 & b-12\end{pmatrix}$$

So the system has infinite solution when:

$$\boldsymbol{a=5, b=12}$$

(25 pts)

3. Let $B=\left\{\bar{v}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}, \bar{v}_2=\begin{pmatrix}1\\1\\1\end{pmatrix}, \bar{v}_3=\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ be a basis to $\mathbb{R}^3$, and let T be a linear transformation defined by:

$$T(\bar{v}_1)=\begin{pmatrix}1\\2\\3\end{pmatrix}, T(\bar{v}_2)=\begin{pmatrix}0\\1\\0\end{pmatrix}, T(\bar{v}_3)=\begin{pmatrix}1\\4\\3\end{pmatrix}$$

2.1. Find the matrix representation of T with inputs in the basis B (and outputs in the standard basis).

The columns of the matrix are the images of the corresponding basis vectors:

$$A_B=\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 4\\ 3 & 0 & 3\end{pmatrix}$$

2.2. Find the matrix representation of T in the standard basis. [7 pts]

Using the basis vectors of B, we can easily find the image of the basis vectors in the standard basis:

$$A_S\hat{e}_1=A_S(\bar{v}_3)=\begin{pmatrix}1\\4\\3\end{pmatrix}$$

$$A_S\hat{e}_2=A_S(\bar{v}_2-\bar{v}_1)=\begin{pmatrix}0\\1\\0\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\-1\\-3\end{pmatrix}$$

$$A_S\hat{e}_3=A_S(\bar{v}_1-\bar{v}_3)=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}0\\-2\\0\end{pmatrix}$$

Therefore:

$$A_S = \begin{pmatrix} 1 & -1 & 0 \\ 4 & -1 & -2 \\ 3 & -3 & 0 \end{pmatrix}$$

2.3. Find a basis for $Im(T)$. [10 pts]

To find a basis of $Im(T)$, we can row-reduce A^T (this will allow us to find a basis for the rows of A^T, which correspond to the columns of A):

$$\begin{pmatrix} 1 & 4 & 3 \\ -1 & -1 & -3 \\ 0 & -2 & 0 \end{pmatrix} R_2 + R_1 \to \begin{pmatrix} 1 & 4 & 3 \\ 0 & 3 & 0 \\ 0 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using the two pivots, a basis for $Im(A)$ is:

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2.4. Find the dimension of $\ker(T)$. [3 pts]

Since $\dim\big(Im(T)\big) = 2$, and the domain has dimension 3, $\dim(\ker(T)) = 1$.

2.5. Is T invertible? [2 pts]

T is not invertible, as it has a non-trivial kernel.

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
If it is true, explain or prove shortly. If it is false, give a counter example.
(answers without a proper explanation will not get any points)

1. Let $A \in M_{n\times n}(\mathbb{R})$. If A is diagonalizable, then so is A^2.

True. For any eigenvector $\bar{v}$ with eigenvalue λ we have $A^2\bar{v} = A(A\bar{v}) = \lambda A\bar{v} = \lambda^2\bar{v}$. So if A is diagonalizable and has n independent eigenvectors, so does A^2.

2. Every diagonalizable matrix $A \in M_{n\times n}(\mathbb{R})$ is also invertible.

False. The zero matrix is diagonalizable (in fact, it is already diagonal), but it is not invertible.

3. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

True. For any eigenvector $\bar{v}$ with eigenvalue λ we have $A^{\mathrm{k}}\bar{v} = \Big(\underbrace{AAA \dots A}_{k\ times}\Big)\bar{v} = \lambda^k\bar{v}$.

4. If A is a real $n \times n$ matrix with linearly independent columns, then there is always a solution to $A\bar{x} = \bar{b}$.

True. If A has independent columns, then it is regular (A^{-1} exists), so there is a unique solution for any $\bar{b}$.

Another answer- If A has independent columns and its square (n x n) so im(A)=R^n. then any b in R^n is in the image.

5. If $A^2 = 0$ then also $A = 0$.

False. Let's take a matrix that sends $\hat{\imath}$ to zero, and $\hat{\jmath}$ to $\hat{\imath}$. Then:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(5 pts each)

Good luck to all of you!