(2) (a) We can write V as follows:

$$V = \left\{ \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

From this description it is clear that $\dim V = 3$, and B can be:

$$B = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(b) The columns of $[T]_E^E$ are $[Te_1]_E, [Te_2]_E$ and $[Te_3]_E$, when e_1, e_2, e_3 are the vectors of the standard basis of $\mathbb{R}^3$, and the subscripted E refers to the standard basis of $M_2(\mathbb{R})$. We get:

$$[T]_E^E = \begin{bmatrix} 0 & 0 & 0 \\ m+1 & 2 & m-1 \\ 0 & m^2 & -1 \\ 0 & 1-m^2 & 0 \end{bmatrix}$$

Since the $_{11}$ element in $T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is 0, it is clear that $\mathrm{Im}(T) \subseteq V$. For equality to hold between the subspaces, they must be of the same dimension. We will therefore check for which values if m the rank of $[T]_E^E$ is 3 (and in particular we will conclude for the dimension of the image, since it is the dimension of the column space). Observing the 4th row, it is clear that if $m = 1$ or $m = -1$ the rank of the matrix is smaller than 3. Furthermore, if $m \neq 0$ we may subtract from the fourth row $\frac{1-m^2}{m^2}$ times the third row to get the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ m+1 & 2 & m-1 \\ 0 & m^2 & -1 \\ 0 & 1-m^2 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - \frac{1-m^2}{m^2} R_3} \begin{bmatrix} 0 & 0 & 0 \\ m+1 & 2 & m-1 \\ 0 & m^2 & -1 \\ 0 & 0 & \frac{1-m^2}{m^2} \end{bmatrix}$$

This matrix is of rank 3 if $m \neq \pm 1$. The last case, $m = 0$, also gives a rank 3 martix. So $\mathrm{Im}(T) = V$ if and only if $m \neq 1$ and $m \neq -1$.

(c) We will apply the dimension theorem for linear transformations:

$$\dim \mathbb{R}^3 = \dim \mathrm{Im}(T) + \dim \ker(T)$$

We therefore need to know $\dim \mathrm{Im}(T)$ for the different cases of m. In the previous section we saw that when $m \neq 1$ and $m \neq -1$ $\dim \mathrm{Im}(T) = 3$, so $\dim \ker(T) = 0$. When $m = 1$ the representation matrix becomes

$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which is of rank 2, so in this case $\dim \ker(T) = 1$. When $m = -1$ the representation matrix becomes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which is of rank 1, so in this case $\dim \ker(T) = 2$.