Supplementary Math Course -Linear Algebra (76967)

Final Exam (semester A – MOED A) 7.4.2024

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations:

$$x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12$$

   a) Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]
   b) Find det(A). Is A invertible? [6 pt]
   c) Without any further calculations, what can you say about the number of solutions of the system? [8 pt]
   d) Use elementary row operations to either find the solution/s of the system or determine if there is no solution. [10 pt]

(25 pts)

2. Let $A = \begin{bmatrix} a & -1 \\ 1 & 4 \end{bmatrix}$, where a is some real number.
   Suppose that the matrix A has an eigenvalue 3.
   e) Determine the value of a. [6 pts]
   f) Does the matrix A have eigenvalues other than 3? [7 pts]
   g) Find the eigenvectors of each eigenvalue. [7 pts]
   h) What is the algebraic and geometric multiplicity of each eigenvalue? [5 pts]

(25 pts)

3. Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$
   a) Verify that the vectors $\left\{\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ are eigenvectors of the linear transformation T. [5 pts]
   b) Find the matrix representation of T in the basis $\{\bar{v}_1, \bar{v}_2\}$. [5 pts]
   c) Find the matrix representation of T in the standard basis. [5 pts]
   d) What is $rank(T)$? Explain. Does the answer depend on the choice of the basis? [5 pts]
   e) What is $ker(T)$? Explain. Does the answer depend on the choice of the basis? [5 pts]

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. $\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\6\\-2\end{bmatrix}\}$ is a basis for Span(S) where S=$\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\6\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}\}$.

2. Let A and B be n×n matrices (where n is an integer greater than 1). then det(A+B) =det(A)+det(B).

3. for any value of a the set S=$\{\begin{bmatrix}1\\2\\3\\a\end{bmatrix}, \begin{bmatrix}a\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\a^2\\7\end{bmatrix}, \begin{bmatrix}1\\a\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\3\\a^3\end{bmatrix}\}$ is linearly dependent.

4. let A be a square invertible matrix. If $A^2 = A$, then A is the identity matrix.

5. Let A and B be n×n matrices. then $\ker(A) \cap \ker(B) \subset \ker(A+B)$.

(5 pts each)

Good luck to all of you!