Supplementary Math Course -Linear Algebra (76967)

Final Exam (semester A – MOED A) 7.4.2024- **solution**

Part A

Answer all **3** questions.

Next to each question is an estimate of the number of points it is worth.

1. Consider the following system of equations:

$$x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12$$

a. Write down the matrix form of the system ($A\bar{x} = \bar{b}$). [1 pt]

The matrix form of the system ($A\bar{x} = \bar{b}$):

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

b. Find det(A). Is A invertible? [6 pt]

det(A) = 0. Since the determinant is zero, matrix A is not invertible.

c. Without any further calculations, what can you say about the number of solutions of the system? [8 pt]

Since the determinant of the matrix A is zero, it implies that the system of equations represented by A has no unique solution. The system can have either infinitely many solutions or no solutions at all.

d. Use elementary row operations to either find the solution/s of the system or determine if there is no solution. [10 pt]

First, the augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right].$$

We apply elementary row operations as follows to reduce the system to row echelon form.

$$A \xrightarrow{R_3 - 9R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & -8 & -16 & -24 \end{array}\right] \xrightarrow{-\frac{1}{8}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in row echelon form.

The corresponding system of linear equations of it is

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + 2z &= 3 \\ 0z &= 0 \end{aligned}$$

The last equation $0z = 0$ means that z can be any number.
(More systematically, the variables corresponding to leading 1's in the echelon form matrix are dependent variables, and the rests are independent (free) variables.)

So let us say that t is a value for z, namely $z = t$.
Then from the second equation, we have $y = -2t + 3$.
From the first equation, we have

$$x = -2y - 3z + 4 = -2(-2t + 3) - 3t + 4 = t - 2.$$

Thus the solution set is

$$(x, y, z) = (t - 2, -2t + 3, t)$$

for any t.

the system has infinitely many solutions.

(25 pts)

2. Let $A = \begin{bmatrix} a & -1 \\ 1 & 4 \end{bmatrix}$, where a is some real number.
Suppose that the matrix A has an eigenvalue 3.

a. Determine the value of a. [6 pts]

Since 3 is an eigenvalue of the matrix A, we have

$$0 = \det(A - 3I),$$

where I is the 2×2 identity matrix.
Thus we have

$$\begin{aligned} 0 &= \det(A - 3I) \\ &= \begin{vmatrix} a - 3 & -1 \\ 1 & 4 - 3 \end{vmatrix} \\ &= \begin{vmatrix} a - 3 & -1 \\ 1 & 1 \end{vmatrix} \\ &= (a - 3)(1) - (-1)(1) = a - 2. \end{aligned}$$

Thus the value of a must be 2.

b. Does the matrix A have eigenvalues other than 3? [7 pts]

Let us determine all the eigenvalue of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

We compute the characteristic polynomial $p(t) = \det(A - tI)$ of A.

We have

$$\begin{aligned} p(t) &= \det(A - tI) \\ &= \begin{vmatrix} 2-t & -1 \\ 1 & 4-t \end{vmatrix} \\ &= (2-t)(4-t) - (-1)(1) \\ &= t^2 - 6t + 9 \\ &= (t-3)^2. \end{aligned}$$

Since the eigenvalues are roots of the characteristic polynomial, solving $(t-3)^2 = 0$ we see that $t = 3$ is the only eigenvalue of A (with algebraic multiplicity 2).

Hence the matrix A does not have eigenvalues other than 3.

c. Find the eigenvectors of each eigenvalue. [7 pts]

Now, let's find the eigenvectors corresponding to $\lambda = 3$ by solving:

$$(A - 3I)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, we have:

$-v_1 - v_2 = 0$

$v_2 = -v_1$

So, the eigenvector corresponding to $\lambda = 3$ can be any non-zero multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence, the eigenvalue is $\lambda = 3$ with corresponding eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

d. What is the algebraic and geometric multiplicity of each eigenvalue? [5 pts]

The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial, counting repeated roots according to their multiplicity.

In our case, the characteristic polynomial is $\lambda^2 - 6\lambda + 9$, and it factors as $(\lambda - 3)^2$. So, the eigenvalue $\lambda = 3$ appears twice as a root of the characteristic polynomial.

Therefore, the algebraic multiplicity of the eigenvalue $\lambda = 3$ is 2.

The geometric multiplicity of an eigenvalue is the dimension of the eigenspace corresponding to that eigenvalue. In other words, it represents the number of linearly independent eigenvectors associated with that eigenvalue.

For the eigenvalue $\lambda = 3$, we found the corresponding eigenvector to be $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This eigenvector spans a 1-dimensional subspace.

Since the geometric multiplicity represents the dimension of the eigenspace, and we have found only one linearly independent eigenvector for $\lambda = 3$, the geometric multiplicity of $\lambda = 3$ is 1.

In summary:

- Eigenvalue $\lambda = 3$:
  - Algebraic multiplicity: 2
  - Geometric multiplicity: 1

(25 pts)

3. Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$

a) Verify that the vectors $\left\{\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ are eigenvectors of the linear transformation T. [5 pts]

We compute that

$$T(\mathbf{v}_1) = T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T(\mathbf{v}_2) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\mathbf{v}_2.$$

Thus, $\mathbf{v}_1$ is an eigenvector corresponding to the eigenvalue 2 and $\mathbf{v}_2$ is an eigenvector corresponding to the eigenvalue 4.

Since $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues, they are linearly independent, and thus $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$.

b) Find the matrix representation of T in the basis $\{\bar{v}_1, \bar{v}_2\}$. [5 pts]

From the computation in part (a), we have

$$T(\mathbf{v}_1) = 2\mathbf{v}_1 + 0\mathbf{v}_2$$
$$T(\mathbf{v}_2) = 0\mathbf{v}_1 + 4\mathbf{v}_2.$$

Hence the coordinate vectors of $T(\mathbf{v}_1), T(\mathbf{v}_2)$ with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$ are

$$[T(\mathbf{v}_1)]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [T(\mathbf{v}_2)]_B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Thus the matrix A of the linear transformation T with respect to the basis B is

$$A = [\,[T(\mathbf{v}_1)]_B, [T(\mathbf{v}_2)]_B\,] = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

c) Find the matrix representation of T in the standard basis. [5 pts]

To find the matrix representation of the linear transformation T in the standard basis, we consider how T acts on the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of $\mathbb{R}^2$.

The standard basis vectors are:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We apply the transformation T to these vectors:

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(1) + 0 \\ 1(1) + 3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(0) + 1 \\ 1(0) + 3(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now, we arrange the resulting vectors as columns of the matrix representation of T:

$$[T] = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

d) What is $rank(T)$? Explain. Does the answer depend on the choice of the basis? [5 pts]

The rank of a linear transformation is the dimension of the image (or range) of the transformation. In this case, the transformation matrix T is:

$$T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

To find the rank of this matrix, we can use various methods such as row reduction or examining its determinant. For this matrix, it's apparent that both rows are linearly independent, so the rank of T is 2.

Regarding whether the rank changes with the basis: No, the rank of a linear transformation does not depend on the choice of basis. It is an intrinsic property of the transformation itself. Different bases may lead to different representations of the transformation, but the rank remains the same.

In this particular case, whether we express the transformation in the standard basis or the basis formed by $\mathbf{v}_1$ and $\mathbf{v}_2$, the rank of the transformation remains 2.

e) What is $ker(T)$? Explain. Does the answer depend on the choice of the basis? [5 pts]

The kernel of a linear transformation, also known as the null space, consists of all the vectors in the domain that are mapped to the zero vector in the codomain.

For the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$, we want to find the kernel of T.

To find the kernel, we need to solve the equation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, we have:

$$\begin{cases} 3x_1 + x_2 = 0 \\ x_1 + 3x_2 = 0 \end{cases}$$

Solving this system of equations, we get:

$$x_1 = 0 \quad \text{and} \quad x_2 = 0$$

This implies that the only vector in the domain $\mathbb{R}^2$ that gets mapped to the zero vector in the codomain $\mathbb{R}^2$ is the zero vector itself.

Therefore, the kernel of the transformation T is the set containing only the zero vector:

$$\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

The kernel of a linear transformation is an intrinsic property of the transformation itself and does not depend on the choice of basis. It represents the set of vectors in the domain that are mapped to the zero vector in the codomain, regardless of how the transformation is represented in different bases.

So, whether we express the transformation in the standard basis or any other basis, the kernel of the linear transformation remains the same. It is solely determined by the transformation's properties, not by the basis used to represent it.

(25 pts)

Part B

For each of the following statements, determine if it is true or false.
Explain / prove shortly / give a counter example when needed.
(answers without a proper explanation will not get any points)

1. $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\6\\-2\end{bmatrix}\right\}$ is a basis for Span(S) where S=$\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\6\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$

true. Lets find the basis for S:

We will first use the leading 1 method. Consider the system

$$x_1\begin{bmatrix}1\\2\\1\end{bmatrix}+x_2\begin{bmatrix}-1\\-2\\-1\end{bmatrix}+x_3\begin{bmatrix}2\\6\\-2\end{bmatrix}+x_4\begin{bmatrix}1\\1\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c}1&-1&2&1&0\\2&-2&6&1&0\\1&-1&-2&3&0\end{array}\right]\xrightarrow[R_3-R_1]{R_2-2R_1}\left[\begin{array}{cccc|c}1&-1&2&1&0\\0&0&2&-1&0\\0&0&-4&2&0\end{array}\right]$$

$$\to\left[\begin{array}{cccc|c}1&-1&0&2&0\\0&0&1&-1/2&0\\0&0&0&0&\end{array}\right].$$

Since the above matrix has leading 1's in the first and third columns, we can conclude that the first and third vectors of S form a basis of Span(S). Thus

$$\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\6\\-2\end{bmatrix}\right\}$$

is a basis for Span(S).

2. Let A and B be n×n matrices (where n is an integer greater than 1). then det(A+B) =det(A)+det(B).

We claim that the statement is false.

As a counterexample, consider the matrices

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix} \text{ and } B=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

Then

$$A+B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

and we have

$$\det(A+B)=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1.$$

On the other hand, the determinants of A and B are

$$\det(A)=0 \text{ and } \det(B)=0,$$

and hence

$$\det(A)+\det(B)=0\neq 1=\det(A+B).$$

Therefore, the statement is false and in general we have

$$\det(A+B)\neq\det(A)+\det(B).$$

3. for any value of a the set S=$\left\{\begin{bmatrix}1\\2\\3\\a\end{bmatrix}, \begin{bmatrix}a\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\a^2\\7\end{bmatrix}, \begin{bmatrix}1\\a\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\3\\a^3\end{bmatrix}\right\}$ is linearly dependent.

true.

Since the set S consists of five 4-dimensional vectors, it is linearly dependent regardless of the value of a. Thus, for any value of a the set S is linearly dependent.

4. let A be a square invertible matrix. If $A^2 = A$, then A is the identity matrix.

true.

Let A be an $n \times n$ invertible idempotent matrix.

Since A is invertible, the inverse matrix A^{-1} of A exists and it satisfies $A^{-1}A = I_n$, where I_n is the $n \times n$ identity matrix.

Since A is idempotent, we have $A^2 = A$.
Multiplying this equality by A^{-1} from the left, we get $A^{-1}A^2 = A^{-1}A$. Using the fact that $A^{-1}A = I_n$, we obtain $A = I_n$.

The proof is completed.

5. Let A and B be n×n matrices. then $\ker(A) \cap \ker(B) \subset \ker(A+B)$.

True.

Let $\mathbf{x}$ be an arbitrary vector in the intersection $\mathcal{N}(A) \cap \mathcal{N}(B)$.
Then the vector $\mathbf{x}$ belongs to both $\mathcal{N}(A)$ and $\mathcal{N}(B)$.
Thus, by definition of the null space, we have

$$A\mathbf{x} = \mathbf{0} \text{ and } B\mathbf{x} = \mathbf{0}.$$

If follows from these equalities that we have

$$(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence $\mathbf{x}$ lies in the null space of the matrix $A+B$, that is, $\mathbf{x} \in \mathcal{N}(A+B)$.

Since $\mathbf{x}$ is an arbitrary element of $\mathcal{N}(A) \cap \mathcal{N}(B)$, we have shown the inclusion

$$\mathcal{N}(A) \cap \mathcal{N}(B) \subset \mathcal{N}(A+B),$$

as required.

(5 pts each)

Good luck to all of you!