THE MINIMAL MODEL PROGRAM
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Abstract. The goal of this course is to provide an introduction to the minimal model program which aims to study the birational geometry of varieties by constructing for each birational equivalence class a representative which is minimal or canonical in a certain sense.

It is a classical result that for non-rational surfaces there always exists a minimal smooth representative, which can be constructed by first resolving singularities of an arbitrary representative, and then successively blowing down exceptional curves. For rational surfaces, however, this uniqueness fails, and the different minimal models are connected by so-called Sarkisov links.

Already the case of threefolds is tremendously more difficult – one is forced to allow mildly singular minimal models and in general there exist many different minimal models connected by flops. In addition to blowing down divisors and flops, one also has to consider much finer contractions, called flips, which only alter the geometry in codimension 2 but improve the effective cone of curves on the model, etc.

In this course we will give an introduction to these methods, along the lines of the book of Kollár and Mori.

If time permits, we will also discuss some of the more recent great achievements:

1) The construction of canonical models by Hacon, McKernan and others. 2) The boundedness of Fano varieties by recent Fields medalist Birkar. 3) The homological minimal model program which seeks to understand birational geometry through derived categories of sheaves and their semiorthogonal decompositions.

1. Lecture 1:The Enriques classification of surfaces from the viewpoint of MMP

We begin this lecture series by presenting a classical result from the viewpoint of the minimal model program. Our goal in this first lecture will be Enriques classification of surfaces:

Theorem 1.1. *Let S be a nonsingular projective surface over an algebraically closed field k of characteristic zero. Then the birational classification of S is given by the following table: Here*

$\kappa(S)$	p_g	q	*a birational representative*
$-\infty$	0	0	$\mathbb{P}^2$
$-\infty$	0	>0	$\mathbb{P}^1 \times C$, *C a smooth curve of genus q*
0	1	0	*a K3 surface*
0	0	0	*an Enriques surface, i.e. the quotient of a K3 surface by a fixed-point-free involution*
0	1	2	*an Abelian surface*
0	0	1	*a bielliptic surface, i.e. the étale quotient of a product of two elliptic curves*
1	$\geqslant 0$	$\geqslant 0$	*an elliptic surface, i.e. a surface with an elliptic fibration over a smooth curve*
2	$\geqslant 0$	$\geqslant 0$	*a surface of general type*

Table 1. The table of exceptional counterexamples

$\kappa(S)$ is the Kodaira dimension of S, which will be explained shortly, and

$$p_g = p_g(S) = h^0(S, \mathcal{O}_S(K_S)), \quad \textit{the geometric genus of } S$$

$$q = q(S) = h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S), \quad \textit{the irregularity of } S.$$

All of these integers are birational invariants.

The minimal model program approaches this classification using the following (vaguely stated) algorithm:

(1) Find a good representative in a given birational equivalence class.
(2) Study the properties of the good representative.
(3) Study the (birational) relation among possibly many choices of the good representatives.
(4) Construct the moduli space of these varities, fixing some discrete invariants like genus or Chern classes but varying the birational equivalence classes.

For the most part, we will stay within the realm of birational geometry and stick to Steps 1-3 in these lectures.

1.1. **Preliminaries on Cones of curves.**

Definition 1.2. Let X be a proper variety. A *1-cycle*is a formal linear combination of integral proper curves $C = \sum a_i C_i$ with real, rational, or integral coefficients. A 1-cycle is *effective* if $a_i \geqslant 0$ for every i. Two 1-cycles C, C' are called *numerically equivalent* if $(C \cdot D) = (C' \cdot D)$ for any Cartier divisor D. The space of real 1-cycles modulo numerical equivalence form the $\mathbb{R}$-vector space we denote by $N_1(X)$. The numerical class of a 1-cycle C is denoted by $[C]$.

Recall now that the Néron-Severi group $\mathrm{NS}(X)$ of X is the set of divisors on X modulo algebraic equivalence and by [1, Exercises V.1.7] this gives a perfect pairing

$$N_1(X) \times \mathrm{NS}(X)_{\mathbb{R}} \to \mathbb{R}.$$

By the Theorem of the Base of Néron-Severi, this duality is between finite dimenisonal $\mathbb{R}$-vector spaces whose dimension is called the *Picard rank* of X and denoted by $\rho(X)$. In characteristic zero this can be proved easily as in [1, Exercise V.1.8] or when over $\mathbb{C}$ using the exponential short exact sequence and the Lefschetz $(1,1)$-theorem.

Definition 1.3. Let X be a proper variety. Set

$$NE_{\mathbb{Q}}(X) = \{\sum a_i[C_i] : C_i \subset X, 0 \leqslant a_i \in \mathbb{Q}\} \subset N_1(X);$$

$$NE(X) = \{\sum a_i[C_i] : C_i \subset X, 0 \leqslant a_i \in \mathbb{R}\} \subset N_1(X);$$

$$\overline{NE}(X) = \text{the closure of } NE(X) \text{ in } N_1(X),$$

where the C_i are proper, integral curves of X. clearly $NE_{\mathbb{Q}}(X)$ is dense in $\overline{NE}(X)$. The convex cone $\overline{NE}(X)$ is called *the cone of (effective) curves* of X.

For any numerical divisor D, set $D_{\geqslant 0} := \{x \in N_1(X) : (x \cdot D) \geqslant 0\}$ (and similarly for > 0, $\leqslant 0$, < 0) and set $D^{\perp} := \{x : (x \cdot D) = 0\}$. We also use the notation

$$\overline{NE}(X)_{D \geqslant 0} := \overline{NE}(X) \cap D_{\geqslant 0},$$

and similarly for > 0, $\leqslant 0$, and < 0.

Theorem 1.4 (Kleiman's Ampleness Criterion)**.** *Let X be a projective variety and D a Cartier divisor on X. Then D is ample iff*

$$D_{>0} \supset \overline{NE}(X) \backslash \{0\}.$$

Corollary 1.5. *Let X be a projective variety and H an ample divisor. Then:*

(1) *$\overline{NE}(X)$ does not contain a straight line.*
(2) *For any constant $C > 0$ the set $\{z \in \overline{NE}(X) : (z \cdot H) \leqslant C\}$ is compact.*
(3) *For any constant $C > 0$ there are only finitely many numerical equivalence classes of effective 1-cycles $Z = \sum a_i Z_i$ with integral coefficients such that $(Z \cdot H) \leqslant C$.*

Proof. To prove (1), we note that intersection with H defines a linear functional on $N_1(X)$ which is positive on $\overline{NE}(X)\backslash\{0\}$. But a linear functional cannot possibly be positive on a straight line minus the origin, so $\overline{NE}(X)$ cannot contain a straight line.

To see the other claims, fix a norm $\| \quad \|$ on $N_1(X)$ and assume to the contrary that $W_C := z \in \overline{NE}(X) : (z \cdot H) \leqslant C\}$ is not compact. Then there is a sequence $z_i \in W_C$ such that $\|z_i\| \to \infty$. But $z_i/\|z_i\|$ is a bounded sequence hence a suitable subsequence converges to a point $y \in \overline{NE}(X)\backslash\{0\}$. But as $(y \cdot H) = \lim \frac{(z_i \cdot H)}{\|z_i\|} = 0$, this contradicts H being ample by Kleiman's ampleness criterion, so W_C is indeed compact.

Finally, note that integral 1-cycles correspond to a discrete set in $N_1(X)$, so it has only finitely many points in any compact set. □

The significance of the cone of curves is the following foundational result of Mori which holds in the nonsingular case in arbitrary dimension.

Theorem 1.6 (The Cone Theorem [5])**.** *Let X be a nonsingular projective variety.*

(1) *There are countably many rational curves $C_i \subset X$ such that*

$$0 < -(C_i \cdot K_X) \leqslant \dim X + 1, \quad \text{and} \quad \overline{NE}(X) = \overline{NE}(X)_{K_X \geqslant 0} + \sum_i \mathbb{R}_{\geqslant 0}[C_i].$$

(2) *For any $\epsilon > 0$ and ample divisor H, of*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \epsilon H) \geqslant 0} + \sum_{\text{finite}} \mathbb{R}_{\geqslant 0}[C_i].$$

Proof. For the proof see [2, Theorem 1.28]. We'll circle back to the proof and its fundamental ideas later in the course. □

While the Cone Theorem reveals interesting structure of the cone of curves, the relation with birational and classical geometry is provided by realization that contractions correspond to extremal faces of this cone. Recall that a subcone $M \subset N$ of a cone N is called *extremal* or *an extremal face of* N if M satisfies the property that $u, v \in N$ and $u + v \in M$ imply that $u, v \in M$. A 1-dimensional extremal subcone is called an *extremal ray*.

Definition 1.7. Let X be a projective variety and $F \subset \overline{NE}(X)$ an extremal face. A morphism $\mathrm{cont}_F \colon X \to Z$ to a normal projective variety Z is called the *contraction* of F if the following hold:

(1) $\mathrm{cont}_F(C) = \text{point}$ for an irreducible curve $C \subset X$ iff $[C] \in F$;
(2) $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$.

A contraction of a K_X-negative extremal face with $\dim Z < \dim X$ is called a *Mori fiber space.*

Remark 1.8. A contraction being a Mori fiber space is equivalent to the condition (i) it being a morphism with connected fibers onto a normal projective variety of smaller dimension along with the condition (ii) that all irreducible curves C in a fiber are numerically proportional with $(K_X \cdot C) < 0$.

Remark 1.9. In general not every extremal face can be contracted [2, Example 1.27], and it is not clear when cont_F exists. Nevertheless, these two conditions uniquely define cont_F. Indeed, the second condition specifies that cont_F is its own Stein factorization. In particular, the fibers are connected, so the first condition specifies the fibers of cont_F set-theoretically. The Stein factorization from the second condition guarantees that this specifies cont_F as a morphism as well.

Conversely, let $g \colon X \to Z$ be a projective morphism such that $g_* \mathcal{O}_X \to Z$. Let F be the closed cone spanned by all $[C]$ such that the integral curve $C \subset X$ is sent ot a poitn by g. Then $\mathrm{cont}_F = g$, so g is a contraction of an extremal face.

Exercise 1.10. Prove why the F above is indeed extremal.

1.2. **Step 1: Running the algorithm for surfaces.**

1.2.1. *Cone of curves in the surface case.*

Theorem 1.11 (Riemann-Roch for surfaces)**.** *For a divisor D on a nonsingular projective surface S, then*

$$\sum_i (-1)^i h^i(S, \mathcal{O}_S(D)) = \chi(\mathcal{O}_S(D)) = \frac{1}{2}(D - K_S)\cdot D + \frac{1}{12}\Big((-K_S)^2 + e(S)\Big),$$

where $e(S)$ is the topological Euler characteristic. Here we have use the Noether formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}\Big((-K_S)^2 + e(S)\Big).$$

Lemma 1.12. *If D is a divisor on an integral, proper surface X with $(D^2) > 0$, then for $n \gg 0$ either $|nD| \neq \varnothing$ or $|-nD| \neq \varnothing$.*

Proof. By Riemann-Roch and Serre duality,

$$h^0(nD) + h^0(K_X - nD) \geqslant \chi(nD) = \frac{n^2}{2}(D^2) - \frac{n}{2}(D\cdot K_X) + \chi(\mathcal{O}_X),$$

$$h^0(-nD) + h^0(K_X + nD) \geqslant \chi(nD) = \frac{n^2}{2}(D^2) + \frac{n}{2}(D\cdot K_X) + \chi(\mathcal{O}_X).$$

As $n \to \infty$, the right-hand side of both inequalities grows quadratically with n. But for $n \gg 0$, we cannot have both $h^0(K_X - nD)$ and $h^0(K_X + nD)$ growing large as the two divisors sum to a fixed linear system $|2K_X|$. Thus $h^0(nD)$ or $h^0(-nD)$ grows quadratically with n. ☐

Corollary 1.13. *Let X be an integral, projective surface with an ample divisor H. The set $Q := \{z \in N_1(X) : (z^2) > 0\}$ has two connected components*

$$Q^+ := \{z \in Q : (z\cdot H) > 0\} \qquad \textit{and} \qquad Q^- := \{z \in Q : (z\cdot H) < 0\}.$$

Furthermore, $Q^+ \subset \overline{NE}(X)$.

Proof. By the Hodge-Index Theorem the intersection form on $N_1(X)$ has signature $(1, \rho(X) - 1)$ (see [1, Remark V.1.9.1], so in a suitable basis we may write it as $x_1^2 - \sum_{i\geqslant 2} x_i^2$. We can even choose the basis such that $[H]$ has coordinates $\left(\sqrt{(H^2)}, 0, \ldots, 0\right)$. This gives the two connected component

$$Q^+ = \left(x_1 > \sqrt{\sum_{i\geqslant 2} x_i^2}\right) \qquad \text{and} \qquad Q^- = \left(x_1 < -\sqrt{\sum_{i\geqslant 2} x_i^2}\right).$$

For any $[D] \in Q$, by Lemma 1.12 either nD or $-nD$ is effective for $n \gg 0$ and effective curves have positive intersection with H. Thue the effective curves in Q are the ones in Q^+. ☐

Lemma 1.14. *Let X be an integral and projective surface and $C \subset X$ an irreducible curve. If $(C^2) \leqslant 0$, then $[C]$ is in the boundary of $\overline{NE}(X)$. If $(C^2) < 0$ then $[C]$ is extremal in $\overline{NE}(X)$.*

Proof. If $D \subset X$ is an irreducible curve such that $(D\cdot C) < 0$, then $D = C$ since otherwise their intersection number would be non-negative as it counts with multiplicity the number of points in their scheme-theoretic intersection. It follows that

$$\overline{NE}(X) = \mathbb{R}_{\geqslant 0}[C] + \overline{NE}(X)_{C\geqslant 0}.$$

In particular, if $(C^2) = 0$, then intersection with C is a linear functional which is non-negative on $\overline{NE}(X)$ and zero on $[C]$, so it's on the boundary. On the other hand, if $(C^2) < 0$, then $[C] \notin \overline{NE}(X)_{C\geqslant 0}$. Thus $[C]$ generates an extremal ray. ☐

Example 1.15. Let's see some examples of the cone of effective curves on various surfaces:

(1) Let X be a minimal ruled surface over a curve B. Then $\rho(X) = 2$, so $\overline{NE}\,(X)$ is a convex cone in $\mathbb{R}^2$ that does not contain a straight line, so it must be generated by its two edges. Let f be the homology class of the fiber of the $\mathbb{P}^1$ fibration $X \to B$ and denote by s the other edge. We know that f is an edge by the previous lemma and the fact that $f^2 = 0$. If $s^2 < 0$, then take a sequence D_n of effective 1-cycles converging to a point of $\mathbb{R}_{\geqslant 0}s$. Then for $n \gg 0$ we must have $(D_n)^2 < 0$. It follows that there is an irreducible component E_n of $\operatorname{Supp} D_n$ such that $(E_n^2) < 0$. It follows that $[E_n]$ must be extremal by Lemma 1.14. Hence $[E_n] \in \mathbb{R}_{\geqslant 0}s$, so this must be a rational edge as well. If $s^2 = 0$, then take an irreducible fiber D other than the a fiber. Then $[D]$ and f span $N_1(X)$. Writing $s = xf + yD$, we see that

$$0 = (s^2) = (xf + yD)^2 = 2xy(f \cdot D) + y^2(D^2),$$

from which it follows that s is proportional to $(D^2)f - 2(f \cdot D)D$, i.e. $\mathbb{R}_{\geqslant 0}s$ is a rational ray.

In either case, the effective cone is rational and agrees with the closure of the positive cone in the second case.

(2) Let A be an abelian surface with ample divisor H. Since there are no rational curves in an abelian variety, it follows from the adjunction formula that for an irreducible curve $C \subset A$, we must have $C^2 = (C \cdot (C + K_A)) = 2g(C) - 2 \geqslant 0$, so $\overline{NE}\,(A) = \overline{Q^+}$ the closure of the positive cone. If $\rho(A) \geqslant 3$ then $\overline{NE}\,(A)$ is a round cone. This occurs for example if $A \cong E \times E$ for some elliptic curve E (see [1, Exercises IV.4.10,V.1.6]). It follows that every point on the boundary of $\overline{NE}\,(A)$ is extremal. Most of these are irrational and don't correspond to the class of any curve on A.

(3) A classical example is the cubic surface $X \subset \mathbb{P}^3$ which is known to be the blowup of $\mathbb{P}^2$ at six points. Thus $\rho(X) = 7$ and X contains precisely 27 lines $L_1, \ldots, L_{27}$ whose classes generate 27 extremal rays. In fact, one case show that $NE\,(X) = \mathbb{R}_{\geqslant 0}L_1 + \cdots + \mathbb{R}_{\geqslant 0}L_{27}$ so $NE\,(X) = \overline{NE}\,(X)$ is a cone over a finite polyhedron.

1.2.2. *Finding a good birational representative.* So given a proper irreducible surface S over k, let's run this algorithm. Whatever we mean by "good representative", it is natural that it should be projective and as nonsingular as possible. So by normalization, Chow's Lemma [1, Exercise II.4.10], and resolution of singularities, we may suppose that S is a nonsingular projective surface over k. Step 1 of the MMP in the surface case is based on repeated application of the Contraction theorem for surfaces:

Theorem 1.16 (The Contraction Theorem)**.** *Let X be a smooth projective surface and $R \subset \overline{NE}(X)$ an extremal ray such that $(R \cdot K_X) < 0$ then the contraction morphism $\operatorname{cont}_R \colon X \to Z$ exists and is one of the following types:*

(1) *Z is a smooth surface and X is obtained from Z by blowing up a closed point; in this case $\rho(Z) = \rho(X) - 1$.*
(2) *Z is a smooth curve and X is a minimal ruled surface oer Z; in this case $\rho(X) = 2$.*
(3) *Z is a point, $\rho(X) = 1$, and $-K_X$ is ample. (One can even show that $X \cong \mathbb{P}^2$ in this case, but it's harder and not relevant to the main ideas here at the moment.)*

Proof. There is an irreducible curve $C \subset X$ such that $[C] \in R$. We prove that the three cases in the theorem correspond to the sign of (C^2).

Assume first that $(C^2) > 0$. Then $[C]$ must be an interior point of $\overline{NE}(X)$ since it's in the H-positive component $Q^+ = \{z \in \overline{NE}(X) | (z^2) > 0, (H \cdot z) > 0\}$ of the positive cone. But by assumption $[C]$ generates an extremal ray. Thus $N_1(X) \cong \mathbb{R}$. By our further assumption, $(C \cdot K_X) < 0$, thus K_X is negative on $\overline{NE}(X) \backslash \{0\}$. It follows that $-K_X$ is ample by Kleiman's Ampleness criterion Theorem 1.4.

Now consider the case when $(C^2) = 0$. Since C is effective, for any ample divisor H we have $(H \cdot C) > 0$ and thus

$$h^2(X, \mathcal{O}_X(mC)) = h^0(X, \mathcal{O}_X(K_X - mC)) = 0$$

for $m \gg 0$ (for example for $m > \frac{(K_X \cdot H)}{(H \cdot C)}$). It follows that

$$\begin{aligned} h^0(X, \mathcal{O}_X(mC)) &\geqslant h^0\left(X, \mathcal{O}_X\left(mC\right)\right) - h^1\left(X, \mathcal{O}_X\left(mC\right)\right) = \chi\left(X, \mathcal{O}_X\left(mC\right)\right) \\ &= \frac{-(K_X \cdot C)}{2} m + \chi(\mathcal{O}_X) \geqslant 2, \end{aligned}$$

so we can decompose the elements of the linear system $|mC|$ into a fixed part and a moveable part. But as $[C]$ is extremal, any fixed/moveable part is a multiple of C, so there is some $m' \gg 0$ such that $|m'C|$ has no fixed components, that is it's base locus must have codimension at least two. But taking two sections $D, D' \in |m'C|$, we see that they do not intersect as

$$(D \cdot D') = (m')^2(C^2) = 0.$$

Thus $|m'C|$ is base-point free. In this case we let $\operatorname{cont}_R \colon X \to Z$ be the Stein factorization of the morphism corresponding to $|m'C|$.

Let $\sum a_i C_i$ be the decomposition of a fiber of cont_R. Then $\sum a_i[C_i] = [C] \in R$, and since the ray R is extremal, we get $[C_i] \in R$ for every i. Thus $(C_i^2) = 0$ and $(C_i \cdot K_X) < 0$. The adjunction formula tells us that

$$-2 \leqslant 2p_a(C_i) - 2 = (C_i \cdot (C_i + K_X)) = (C_i \cdot K_X) < 0,$$

so that C_i is an irreducible smooth curve isomorphic to $\mathbb{P}^1$ and $(C_i \cdot K_X) = -2$. Thus

$$-2 = (C \cdot K_X) = \left(\sum a_i C_i \cdot K_X\right) = -2 \sum a_i,$$

which shows that every fiber is an irreducible and reduce curve, isomorphic to $\mathbb{P}^1$. It follows that X is a minimal ruled surface over Z.

Finally, assume that $(C^2) < 0$. Then the adjunciton formula tells us that

$$-2 \leqslant 2p_a(C) - 2 = (C \cdot (C + K_X)) = (C^2) + (C \cdot K_X) \leqslant -1 + (C \cdot K_X) < -1,$$

which gives that C is a (-1)-curve, isomorphic to $\mathbb{P}^1$. In this case we let cont_R be the contraction provided by the Castelnuovo contraction theorem. ☐

Theorem 1.17 (Castelnuovo Contraction Theorem,[1, Theorem V.5.7])**.** *If C is a curve on a nonsingular projective surface X with $C \cong \mathbb{P}^1$ and $(C^2) = -1$, then there eists a morphism $f \colon X \to X_0$ to a nonsingular projective surface X_0 and point $p_0 \in X_0$ such that f is the blowup of X_0 at the point p_0 and C is the corresponding exceptional curve.*

Theorem 1.18 ("Running the MMP")**.** *Let X be a smooth projective surface. There is a sequence of contractions $X \to X_1 \to \cdots \to X_n = X'$ such that X' satisfies exactly one of the following conditions:*

1. *$K_{X'}$ is nef;*
2. *X' is a minimal ruled surface over a curve C;*
3. *$X' \cong \mathbb{P}^2$.*

Proof. We start with our smooth projective surface X. If K_X is nef, then we can stop. Otherwise, if K_X is not nef, then by the Cone Theorem we can choose a K_X-negative extremal ray $R \subset \overline{NE}\,(X)$. By the Contraction Theorem, Theorem 1.16, we get a contraction $\operatorname{cont}_R \colon X \to Z$. There are two possibilities. If $\dim X = \dim Z$, then we let $X_1 = Z$ and repeat the proceedure with X_1. As $\rho(X_1) = \rho(X) - 1$, this will eventually terminate. Otherwise, the second two cases of Theorem 1.16 tell us that we may take $X = X'$ as in cases (2) and (3) of this theorem. ☐

Definition 1.19. If X' is nef above, then X' is called a *minimal model* of X. It turns out that in this case the morphism $X \to X'$ is unique, thus X' is determined by X.

The variety X' will be our good representative. In the case of minimal models, we must investigate htese using more refined tools to fill out the Enriques classification.

Exercise 1.20. This exercise will walk you through proving *Castelnuovo's Criterion for Rationality*: Let S be a nonsingular projective surface. Then S is birational to $\mathbb{P}^2$ iff

$$h^1(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(2K_S)) = 0.$$

We note that these are birational invariants. One direction is clear then, since $\mathbb{P}^2$ has these invariants. We break up the reverse implication into steps:

(1) Prove that if $h^1(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(2K_S)) = 0$, then K_S is not nef.
(2) Apply the MMP. What can you get as possible end results?

1.3. **Step 2: Properties of Minimal Models in Dimension 2.**

Definition 1.21. Let S be a nonsingular projective variety. The *Kodaira dimension* $\kappa(S)$ *of* S is defined to be

$$\kappa(S) = \kappa(S, K_S) :=$$
$$\begin{cases} -\infty & \text{if } H^0(S, \mathcal{O}_S(mK_S)) = 0, \quad \forall m \in \mathbb{N}, \\ \left(\text{tran.deg}_{\mathbb{C}} \bigoplus_{m \geqslant 0} H^0\left(S, \mathcal{O}_S\left(mK_S\right)\right)\right) - 1 & \text{if } H^0\left(S, \mathcal{O}_S\left(mK_S\right)\right) \neq 0 \text{ for some } m \in \mathbb{N}. \end{cases}$$

Similarly, for any divisor D we can define $\kappa(S, D)$ and for a singular variety T, the Kodaira dimension is defined to be that of a desingularization.

Proposition 1.22. (1) *The Kodaira dimension is a birational invariant.*
(2) *The Kodaira dimension can alternatively be defined as $-\infty$ or*

$$\max_{m \in \mathbb{N}} \{\dim \Phi_{|mK_S|}(S)\},$$

where $\Phi_{|mK_S|} \colon S \dashrightarrow \mathbb{P}$ is the rational map induced by the linear system $|mK_S|$.
(3) [3, Corollary 2.1.38] *There exist $m_0 \in \mathbb{N}$ and $\alpha, \beta > 0$ such that*

$$\alpha m^{\kappa(S)} \leqslant h^0\left(S, \mathcal{O}_S\left(m m_0 K_S\right)\right) \leqslant \beta m^{\kappa(S)}, \forall m \in \mathbb{N}.$$

(4)

$$\kappa(S) = 0 \Longleftrightarrow h^0\left(S, \mathcal{O}_S\left(mK_S\right)\right) = 0 \text{ or } 1, \quad \forall m \in \mathbb{N}, \text{ and } h^0(S, \mathcal{O}_S(mK_S)) \neq 0 \text{ for some } m \in \mathbb{N}$$

Remark 1.23. From this result it is clear that the Kodaira dimension satisfies

$$\kappa(S) \in \{-\infty, 0, 1, \ldots, \dim S\}$$

Proposition 1.24. *If $\phi \colon X \to Z$ is a Mori fiber space in dimension two, then*

$$H^0(X, \mathcal{O}_X(mK_X)) = 0, \quad \forall m \in \mathbb{N}.$$

That is, $\kappa(X) = -\infty$.

Proof. Suppose to the contrary that for some $m \in \mathbb{N}$ we have an effective $D \in |mK_X| \neq \varnothing$. We can take a curve C that is contained in a fiber of ϕ but not in D.[1] Then

$$0 > (mK_X \cdot C) = (D \cdot C) \geqslant 0,$$

a contradiction. □

[1] Otherwise D would contain all the fibers and thus be all of X.

Proposition 1.25 (Effective Pluricanonical Divisor on Minimal Models in Dim 2)**.** *Let X be a minimal model in dimension two. Then $\kappa(X) \geqslant 0$, i.e.*

$$H^0(X, \mathcal{O}_X(mK_X)) \neq 0 \quad \text{for some } m \in \mathbb{N}.$$

Theorem 1.26 (Hard Dichotomy Theorem of MMP for surfaces)**.** *Let S be a nonsingular projective surface. Then the end result of MMP starting from S is a minimal model (respectively a Mori fiber space) iff $\kappa(S) \geqslant 0$ (respectively $\kappa(S) = -\infty$).*

Proof. This follows from the birational invariance of the Kodaira dimension, Propositions 1.24 and 1.25, and Theorem 1.18. ☐

Proposition 1.27 (Semiample fibrations,[4, Proposition 1-2-16],[3, Theorem 2.1.27])**.** *Let D be a Cartier divisor on a normal projective variety X such that the linear system $|mD|$ is a base-point free for all sufficiently large $m \in \mathbb{N}$. Let $\Phi_{|mD|}\colon X \to Z'$ be the morphism assicoated to $|mD|$, and $X \xrightarrow{\Phi} Z \xrightarrow{\Psi} Z'$ its Stein factorization* [1, Corollary III.11.15]. *Then*

1. *Φ is a morphism with connected fibers onto a normal projective variety Z;*
2. *$(D \cdot C) = 0$ for a curve $C \subset X$ iff C is in a fiber of Φ;*
3. *$\Phi_*(\mathcal{O}_S) = \mathcal{O}_Z$;*
4. *$D = \Phi^* H$ for some ample Cartier divisor H on Z;*
5. *Φ coincides with $\Phi_{|mD|}$ for any sufficiently large $m \in \mathbb{N}$.*

Moreover, Φ is characterized by condition (2) along with either (1) or (3) (they're really equivalent).

This sounds very much like our contraction theorem (of which it is a special case) and we want to apply it to the Cartier divisor K_X. To do so, we need to know that some multiple of K_X is base-point free. This is precisely the statement of the surface case of the so-called Abundance Conjecture for minimal models:

Theorem 1.28 (Abundance Theorem)**.** *Let S be a minimal model in dimension 2. Then $|mK_S|$ is base-point free for sufficiently divisible and large $m \in \mathbb{N}$.*

Theorem 1.29 (Iitaka Fibration in dimension 2)**.** *Let S be a minimal model in dimension 2. There is a morphism*

$$\Phi\colon S \to S_{can}$$

agreeing with $\Phi_{|mK_S|}$ for sufficiently divisible and large $m \in \mathbb{N}$ and such that

1. *Φ is a morphism with connected fibers onto a normal projective variety S_{can},*
2. *for any curve $C \subset S$*
$$\Phi(C) = pt. \Leftrightarrow (K_S \cdot C) = 0,$$
3. *$mK_S = \Phi^*(H)$ for some ample Cartier divisor H on S_{can},*
4. *$\kappa(S) = \dim S_{can}$.*

Properties (1) and (2) characterize $\Phi\colon S \to S_{can}$, which is called the Iitaka fibration *of S onto its* canonical model.

Proof. From the Abundance Theorem it follows that $|mK_S|$ is base-point free for some $m \in \mathbb{N}$. Thus by Proposition 1.27 applied to $D = K_S$ we get the morphism with properties (1)-(3), and (4) follows from Proposition 1.22. ☐

Exercise 1.30. Set $R := \bigoplus_{m \geqslant 0} H^0(S, \mathcal{O}_S(mK_S))$. It has the natrual structure of a graded $\mathbb{C}$-algebra and is called the *canonical ring* of the nonsingular projective surface S. Show that R is finitely generated as a $\mathbb{C}$-algebra and that $S_{can} \cong \operatorname{Proj} R$. Observe that R is a birational invariant, so S_{can} is indeed canonically determined by only the birational equivalence class of the surface.

1.4. **The Enriques classification revisited.** We start with a nonsingular integral projective surface S over $\mathbb{C}$ and run the MMP on S.

1.4.1. $\kappa(S) = -\infty$. If $\kappa(S) = -\infty$, then by Theorem 1.26, the end result will be a Mori fiber space in dimension two $\phi\colon S' \to Z$, i.e. a minimal ruled surface over a smooth curve Z or $\mathbb{P}^2$ (with $Z = \operatorname{Spec}\mathbb{C}$). See [1, Section V.2] for a reveiw, but in the first case, one shows using Tsen's theorem that ϕ has a section which forces it to be isomorphic to $\pi\colon \mathbb{P}(\mathcal{E}) \to Z$ for a rank two vector bundle $\mathcal{E}$ on Z. Trivializing $\mathcal{E}$ over a Zariski open set $U \subset Z$, we see that S' (and thus S) is birational to $Z \times \mathbb{P}^1$, so we may take this as our birational representative in this first case. Observe that

$$p_g(S) = p_g(S') = p_g(Z \times \mathbb{P}^1) = 0, \qquad q(S) = q(S') = q(Z \times \mathbb{P}^1) = g(Z).$$

Thus we may differentiate according to these invariants. Observe that if $q(S) = 0$ as well then $\mathbb{P}^1 \times \mathbb{P}^1$ is indeed birational to $\mathbb{P}^2$, so the two invariants vanishing correspond to the same birational class as required.

1.4.2. $\kappa(S) \geqslant 0$. If $\kappa(S) \geqslant 0$, then running the MMP we must reach a minimal model S_{min} by the Hard Dichotomy Theorem, Theorem 1.26. So we may assume from now on that S is minimal. By the Abundance Theorem, Theorem 1.28, there is a morphism $\Phi\colon S \to S_{can}$ onto the canonical model.

Lemma 1.31 (Kodaira's Lemma)**.** *Let S be a minimal model in dimension two. Then $\kappa(S) = \dim S = 2$ iff $K_S^2 > 0$.*

Proof. Since S is a minimal model, K_S is nef, and thus it is big iff $K_S^2 > 0$. But K_S being big is precisely the condition $\kappa(S) = 2$. □

When $\kappa(S) = 2 = \dim S_{can}$, S is said to be of *general type*. If $K_{S_{min}}$ is nef but not ample, then S_{can} is necessarily singular and Φ is the minimal resolution of singularities. The singularities of canonical models, the so-called *canonical singularities*, can be explicitly classified in dimension two. These are precisely the ordinary double-point singularities with the corresponding ADE classification.

When $\kappa(S) = \dim S_{can} = 1$, generic smoothness [1, Corollary III.10.7] tells us that the general fiber F of $\Phi\colon S_{min} \to S_{can}$ is nonsingular and connected. Moreover, the adjunction formula tells us that

$$g(F) = (K_S + F)\cdot F + 1 = (K_S \cdot F) + (F \cdot F) + 1 = 0 + 0 + 1 = 1.$$

Thus Φ is an elliptic fibration and it can be analyzed further by through Kodaira's study of the possible degenerate fibers of an elliptic fibration. Similarly, the possible multiple fibers can be studied using Kodaira's Canonical Bundle Formula for Elliptic fibrations.

Finally, if $\kappa(S) = \dim S_{can} = 0$, the Iitaka fibration doesn't tell us anything. But note that $K_S^2 = 0$ from Kodaira's Lemma. Moreover, we claim that $\chi(\mathcal{O}_S) \geqslant 0$. Indeed, by Noether's formula

$$12\chi(\mathcal{O}_S) = K_S^2 + e(S) = e(S) = 2(b_0 - b_1) + b_2 = 2(1 - b_1) + b_2.$$

By Serre duality and Hodge theory, however, it follows that

$$\chi(\mathcal{O}_S) = 1 - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - h^1(S, \mathcal{O}_S) + h^0(S, \mathcal{O}_S(K_S)),$$
$$b_1 = h^1(S, \mathbb{C}) = h^1(S, \mathcal{O}_S) + h^0(S, \Omega_S^1) = 2h^1(S, \mathcal{O}_S).$$

Therefore,

$$\begin{aligned} 8\chi(\mathcal{O}_S) &= 12\chi(\mathcal{O}_S) - 4\chi(\mathcal{O}_S) = 2 - 2b_1 + b_2 - 4\left(1 - h^1(S, \mathcal{O}_S) + h^0(S, \mathcal{O}_S(K_S))\right) \\ &= -2 - 2b_1 + 4h^1(S, \mathcal{O}_S) + b_2 - 4h^0(S, \mathcal{O}_S(K_S)) = -2 - 4h^0(S, \mathcal{O}_S(K_S)) + b_2 \\ &\geqslant -2 - 4h^0(S, \mathcal{O}_S(K_S)) \geqslant -6 \end{aligned}$$

since $h^0(S, \mathcal{O}_S(K_S)) \leqslant 1$ by Proposition 1.22. Thus the integer $\chi(\mathcal{O}_S)$ is at least $-\frac{6}{8} = -\frac{3}{4} > -1$, which gives $\chi(\mathcal{O}_S) \geqslant 0$. This puts strong restrictions on the invariants of S:

$$0 \leqslant \chi(\mathcal{O}_S) = 1 - h^1(S, \mathcal{O}_S) + h^0(S, \mathcal{O}_S(K_S)) \leqslant 2 - h^1(S, \mathcal{O}_S).$$

There are five numerical possibilities:

(1) $h^1(S, \mathcal{O}_S) = 2$, $h^0(S, \mathcal{O}_S(K_S)) = 1$, *an Abelian surface.*
(2) $h^1(S, \mathcal{O}_S) = 1$, $h^0(S, \mathcal{O}_S(K_S)) = 1$, *no such surfaces.*
(3) $h^1(S, \mathcal{O}_S) = 1$, $h^0(S, \mathcal{O}_S(K_S)) = 0$, *a bielliptic surface.*
(4) $h^1(S, \mathcal{O}_S) = 0$, $h^0(S, \mathcal{O}_S(K_S)) = 1$, *a K3 surface.*
(5) $h^1(S, \mathcal{O}_S) = 0$, $h^0(S, \mathcal{O}_S(K_S)) = 0$, *an Enriques surface.*

Exercise 1.32. Show that case (2) above really doesn't happen geometrically.

This completes the Enriques classification from the point of view of the Minimal Model Program. To summarize, the Cone and Contraction Theorems let us reduce to two birational classes of particularly simple type $\mathbb{P}^2$ or $\mathbb{P}^1 \times C$ or a minimal model. The Effective Pluricanonical Divisor and Abundance Theorems let us analyze the Iitake fibration to provide extra geometric information except when $\kappa = 0$ which imposes strict numerical constraints.

1.5. **What goes wrong in higher dimension?** Let's suffice our brief look ahead with a look at dimension three and the comment that the Cone theorem still holds in higher dimension. The crucial step in dimension three is then the Contraction Theorem due to Mori:

Theorem 1.33 ([5])**.** *Let X be a nonsingiular projective threefold over $\mathbb{C}$ and $\mathrm{cont}_R \colon X \to Y$ the contraction of a K_X-negative extremal ray $R \subset \overline{NE}(X)$. The following is a list of all possibilities for cont_R:*

(1) ***Exceptional case:*** $\dim Y = 3$*,* cont_R *is biraitonal and there are five types of behavior near the contracted surface:*
  (a) ***E1:***cont_R *is the blowup of a smooth curve in the smooth threefold Y.*
  (b) ***E2:***cont_R *is the blowup of a smooth point in the the smooth threefold Y.*
  (c) ***E3:***cont_R *is the blowup of an ordinary double point of Y (in local analytic coordinates looks like $x^2 + y^2 + z^2 + w^2 = 0$).*
  (d) ***E4:***cont_R *is the blowup of a point of Y locally analytically given by $x^2+y^2+z^2+w^3 = 0$.*
  (e) ***E5:***cont_R *contracts a smooth $\mathbb{P}^2$ with normal point $\mathcal{O}_(-2)$ to a point of multiplicity 4 on Y locally analytically the quotient of $\mathbb{C}^3$ by the involution $(x, y, z) \mapsto (-x, -y, -z)$.*
(2) ***C: (Conic bundles)*** $\dim Y = 2$ *and* cont_R *is a fibration whose fibers are plane conics with smooth general fibers.*
(3) ***D: (Del Pezzo fibtrations):***$\dim Y = 1$ *and general fibers of* cont_R *are Del Pezzo surfaces.*
(4) ***F: (Fano varieties):***$\dim Y = 0$*, $-K_X$ is ample and hence X is a Fano variety.*

Cases C,D,F are nice structure results, especially since the possible smooth Fano threefolds have been completely classified by Iskovkich. The Cases E1 and E2 are the obvious three dimensional analogue of Case (1) of Theorem 1.16, but cases E3, E4, and E5 involve the singular variety Y. To proceed further with the MMP as we would like, we would want to apply Theorem 1.33 again, but it doesn't apply to singular Y. So we're gonna have to deal with singular varieties. The question is how singular?

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