Linear Algebra for Neuroscience (76992)

Exercise 02 – Matrices and linear transformations

1. Let T be the projection transform $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$.

In class we showed that $T(\bar{v} + \bar{u}) = T(\bar{v}) + T\,\bar{u}$.
Complete the proof that to show that T is a linear transformation.

2. Let $T: V \to U$ be a linear transformation. Our definition of linear transformations had two parts.
2.1 Use the first part to prove that any linear transformation maps the zero vector to itself.
2.2 Use the second part to prove that any linear transformation maps the zero vector to itself.

3. The transpose of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order, as rows. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Now, let:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 6 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \bar{v} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}, \bar{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

For all the following products, decide whether the product is well-defined, and if it is – calculate the result:

3.1. $A\bar{v}$
3.2. $D^T\bar{u}$
3.3. AD
3.4. DA
3.5. A^TA
3.6. D^TA

(make sure you understand how the calculation you do relates to the formulas we learned in class for the general term $[A\bar{v}]_i$ and the general term $[AB]_{ij}$.)

3.7 What can you conclude from (3.3) and (3.4) about multiplying a matrix with a diagonal matrix from the left and from the right?

4. Let A be a 3×3 matrix. Find a matrix B such that the i^{th} column of AB is equal to the i^{th} column of A, only multiplied by i. That is, the 1st column of AB is the same as that of A, the 2nd column of AB is the same as the 2nd column of A, only multiplied by 2, and so on.

5. Let T be the transform $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$.

5.1. Describe $Im(T)$ geometrically, in words.
5.2. What is the rank of T?
5.3. Describe $ker(T)$ geometrically, in words.
5.4. What is $\dim(ker(T))$?

5.5. Let A be the matrix that represents T in the standard basis. Write down A.

6. Use the definition of the transpose of a matrix from question 3 and prove that $(AB)^T = B^T A^T$.
Guidance: Denote by a_{ij} and b_{ij} the general entries of A, B. Show that the general entry of $(AB)^T$ is equal to the general entry of $B^T A^T$.

7. Thinking of matrices.
7.1. Find two non-zero matrices A, B such that $AB = 0$.
7.2. Find two matrices A, B such that $AB = 0$ but $BA \neq 0$.
7.3. Find a matrix A such that $AA = A$, but A is not the identity matrix I.

8. Before the next class, please watch videos **6-7** in the "Essence of linear algebra" series:

- "The determinant _ Essence of linear algebra"
- "Inverse matrices, column space and null space"

You can find them both here (or search for the 1blue3brown channel on YouTube):
https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
The next class could start with a short quiz about these videos.

Additional question
The following question is not obligatory.

1. Decide for each of the following transformation if it is linear or not:
1.1. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (xy, x)$
1.2. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x)$
1.3. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
1.4. $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, y + z)$