Question (Ex02, Q6)

Use the definition of the transpose of a matrix from question 3 and prove that $(AB)^T = B^T A^T$.
Guidance: Denote by a_{ij} and b_{ij} the general entries of A, B. Show that the general entry of $(AB)^T$ is equal to the general entry of $B^T A^T$.

Solution

The general term for the ij element in a composite matrix is:

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Therefore, for $(AB)^T$ we will have to switch between the columns and rows and get:

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

Now, for the $B^T A^T$ part, we know that $[B^T]_{ij} = b_{ji}$ and $[A^T]_{ij} = a_{ji}$. So using the general term for the ij element in a composite matrix, we get:

$$[B^T A^T]_{ij} = \sum_{k=1}^{n} [B^T]_{ik} [A^T]_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

Clearly, we found that the two general terms are equal, as we wanted.