1. Let A, P be $n \times n$ matrices, and assume P is regular. Show that for all $m \in \mathbb{N}$:

$$(P^{-1}AP)^m = P^{-1}A^mP$$

Solution

$$(P^{-1}AP)^m = \underbrace{(P^{-1}AP)(P^{-1}AP)\dots(P^{-1}AP)}_{m\ times} = P^{-1}A\underbrace{PP^{-1}}_{I}A\underbrace{PP^{-1}}_{I}\dots\underbrace{PP^{-1}}_{I}AP = P^{-1}A^mP$$

2. Prove by induction that for a regular matrix A and a natural number $m \in \mathbb{N}$:

$$(A^{-1})^m = (A^m)^{-1}$$

Solution

Step 1: Check for $m = 1$:

$$(A^{-1})^1 = (A^1)^{-1}$$

$$A^{-1} = A^{-1}$$

Step 2: Assume for m:

$$(A^{-1})^m = (A^m)^{-1}$$

Step 3: Prove for $m + 1$:

$$(A^{-1})^{m+1} = (A^{-1})^m A^{-1} \underbrace{=}_{using\ step\ 2} (A^m)^{-1}A^{-1}$$

$$(A^{m+1})^{-1} = (AA^m)^{-1} \underbrace{=}_{\substack{(AB)^{-1}= \\ B^{-1}A^{-1}}} (A^m)^{-1}A^{-1} \underbrace{=}_{\substack{from \\ step\ 2}} (A^{-1})^m(A^{-1}) = (A^{-1})^{m+1}$$