Linear Algebra for Neuroscience (76992)

Exercise 03 – Determinant, trace and the inverse matrix

1. Use geometrical terms (and other terms we learned about) to explain the following statement in words:

   If the columns of a square matrix A are linearly dependent, then $\det(A) = 0$.

2. Calculate the determinants of the following matrices. You can use the determinant properties we learned in class or the geometrical interpretation of the linear transformation that each matrix represent:

   2.1. $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$

   2.2. $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   2.3. $\begin{pmatrix} 2 & 7 & -2 & 6 \\ 0 & -3 & 4 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -4 \end{pmatrix}$

   2.4. $\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

3. Let A be a square $(n \times n)$ matrix. Prove the following statements:

   3.1. If $AA^T = I$, then $\det(A) = \pm 1$.

   3.2. If $A = A^2$ and $A \neq I$, then $\det(A) = 0$.

   3.3. If A is regular, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

   3.4. If A is regular, then A^m is regular (for $m \in \mathbb{N}$).

   Note: A square matrix A can be raised to the power of m using the rule:

   $$A^m = \underbrace{AAAAA \dots A}_{m\ times}$$

4. Prove the following statement or give a counter example:

   $$tr(AB) = tr(A)tr(B)$$

5. Let A be a 2×2 matrix with two linearly dependent columns. Given that $tr(A) = 5$:

   5.1. Find $tr(A^2)$

   5.2. Find $\det(A^2)$

6. Matrix powers and the inverse matrix

   6.1. Let A, P be $n \times n$ matrices, and assume P is regular. Show that for all $m \in \mathbb{N}$:

   $$(P^{-1}AP)^m = P^{-1}A^mP$$

   6.2. Prove by induction that for a regular matrix A and a natural number $m \in \mathbb{N}$:

   $$(A^{-1})^m = (A^m)^{-1}$$

Note: A proof by induction is based on three steps:

- Show that the statement holds for $m = 1$
- Assume that it holds for some number $m - 1$
- Use the previous assumption to show that the statement holds for m

If you've never seen a proof by induction, I suggest you read about it here:
https://books.google.co.il/books?redir_esc=y&id=kbooAAAAQBAJ&q=induction#v=snippet&q=induction&f=false
You can also watch an example and explanation in the following video:
https://www.youtube.com/watch?v=CuZJmf3XrTo

7. Definition: A square matrix A is called "nilpotent" if there exists an integer m such that $A^m = 0$ ("nilpotent" means it has the potential to become null, zero). The smallest m for which $A^m = 0$ is called the "nilpotency index" of A.

7.1. What is the nilpotency index of the matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$?

7.2. Can a nilpotent matrix by regular?

7.3. Show that if A is a nilpotent matrix with nilpotency index m, then the matrix $(I - A)$ is regular, with $(I - A)^{-1} = I + A + A^2 + \cdots + A^{m-1}$.
Hint: Since the inverse matrix is unique, you can "guess" what the inverse matrix is and simply show that it satisfies the definition of the inverse matrix. Here you were told what the inverse matrix is, so "guessing" it shouldn't be hard.

Additional question
The following question is not obligatory.

1. Use elementary row operation on A and I (that is, use the augmented matrix $[A|I]$ to find the inverse matrices of:

1.1. $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

1.2. $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$