Question

Prove that if the system $A\bar{x} = \bar{b}$ has more than one solution, then it has infinitely many.
Guidance: Suppose that $\bar{u}$ and $\bar{v}$ are two different solutions. Use them to construct infinite different solutions for the system.

Example solution

Let $\bar{u} \neq \bar{v}$ be two vectors that satisfy the equation:

$$A\bar{v} = A\bar{u} = \bar{b}$$

Let $\bar{w} = \bar{u} - \bar{v}$. We see that:

$$A\bar{w} = A(\bar{u} - \bar{v}) = \bar{b} - \bar{b} = 0$$

Now, let $k \in \mathbb{R}$ be some scalar. So:

$$A(\bar{v} + k\bar{w}) = \bar{b} + 0 = \bar{b}$$

Since k can be any scalar, we found infinite solutions to the equation.

Note 1: Notice that $\bar{w}$ is not the zero vector. It's some vector pointing in some direction.
Note 2: The solution we found agrees with the general case we saw in class – the set of infinite solutions to an inhomogeneous system is made of a particular solution plus a solution to the homogeneous system (in this case, $\bar{w}$ is such a solution).