Linear Algebra for Neuroscience (76992)

Exercise 04 – Systems of linear equations

1. In exercise 1, question 4, you have shown that $\bar{v} = \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix}$ cannot be written as a linear combination of $\bar{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \bar{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \bar{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$.

1.1. Formulate this question as a matrix equation.

1.2. What is the determinant of the matrix whose columns are $\bar{u}_1, \bar{u}_2, \bar{u}_3$? Check your answer by direct calculation.

2. For which values of the parameter a does each of the following system has:

2.1. A unique solution

2.2. Infinite solutions (write down the general solution)

2.3. No solution

Remember: If you divide by a check separately for the case where $a = 0$.

$$\begin{gathered} x + 2y + az = -3 - a \\ x + (2 - a)y - z = 1 - a \\ ax + ay + z = 6 \end{gathered}$$

3. Systems with more variables than equations ($m < n$).

3.1. Solve the following systems of equations. If there exists more than one solution, **find a basis** for the solutions set of the corresponding homogeneous system:

$$\begin{gathered} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 = 7 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 = 7 \end{gathered}$$

3.2. Solve the following systems of equations. If there exists more than one solution, **find a basis** for the solutions set of the corresponding homogeneous system:

$$\begin{gathered} x_1 + 2x_2 - 3x_3 + 4x_4 = 2 \\ 2x_1 + 5x_2 - 2x_3 + x_4 = 1 \\ 5x_1 + 12x_2 - 7x_3 + 6x_4 = 3 \end{gathered}$$

4. True or false?
If a system of linear equations has more variables than equations then it has infinite solutions.

5. Prove that if the system $A\bar{x} = \bar{b}$ has more than one solution, then it has infinitely many.
Guidance: Suppose that $\bar{u}$ and $\bar{v}$ are two different solutions. Use them to construct infinite different solutions for the system.

6. Before the next class, please watch videos **13-14** in the “Essence of linear algebra” series:
    - “Change of basis”
    - “Eigenvectors and eigenvalues”

    You can find them both here (or search for the 1blue3brown channel on YouTube):
    https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
    The next class could start with a short quiz about these videos.

Additional questions

The following questions are not obligatory.

1. Solve the following systems of equations. If there exists more than one solution, write the solution in parametric form:

$$x + 2y - 3z = 1$$
$$2x + 5y - 8z = 4$$
$$3x + 8y - 13z = 7$$

2. Solve the following system of equations. If there exists more than one solution, write the solution in parametric form:

$$x + 2y - 3z = -1$$
$$-3x + y - 2 = -7$$
$$5x + 3y - 4z = 2$$

3. Consider the system:

$$x + ay = 4$$
$$ax + 9y = b$$

3.1. For which value of a does the system have a unique solution?

3.2. For which pairs of values (a, b) does the system have more than one solution? How many solutions does it have then?