Exercise 05 – Eigenvalues and Eigenvectors

Before starting the exercise, please watch video number 13 (its name should be **Eigenvectors and eigenvalues**).

1. Show that if $\bar{v}_1, \bar{v}_2$ are eigenvectors of A that ae associated with the same eigenvalue λ, then $\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2$ is also an eigenvector of A that is associated with λ.

2. In class we said that the trace and determinant are closely related to the eigenvalues of matrix. Let A be a 2×2 matrix, and denote $trace(A) = \tau$ and $\det(A) = \Delta$. Show that in this cases of 2×2 matrices the eigenvalues of A are:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix}$
   3.1. Find the characteristic polynomial of A.
   3.2. Find the eigenvalues of A.
   3.3. Find a basis for the eigenspace of each eigenvalue. How many linearly independent eigenvectors can you find?

4. Let $A = \begin{pmatrix} -9 & 16 & -4 \\ -8 & 15 & -4 \\ -8 & 16 & -5 \end{pmatrix}$
   4.1. Determine whether $\lambda = -1$ is an eigenvalue of A. If it is, find two linearly independent eigenvectors for this eigenvalue.
   4.2. Show that $(1,1,1)^T$ is an eigenvector of A. What is its eigenvalue?

5. Let A be a singular 3×3 matrix. Assume $\det(2I - A) = \det(-2I - A) = 0$.
   5.1. What are the eigenvalues of A?
   5.2. What is the dimension of $ker(A)$?

6. Let $A, B \in M_n(\mathbb{R})$ (i.e, $n \times n$ square matrices with real values). Let λ_1 be an eigenvalue of A and λ_2 be an eigenvalue of B.
   Is the product $\lambda_1\lambda_2$ an always an eigenvalue of AB? It it is, prove. If it isn't, explain in which cases it is and in which cases it is not.

7. Before the next class, please watch videos **13** in the "Essence of linear algebra" series (it should be called "Change of basis"). You can find it here (or search for the 3blue1brown channel on YouTube): https://www.youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=13

The next class could start with a short quiz about this video.

Additional questions

The following questions are not obligatory.

1. Each of the following real matrices defines some linear transformation on $\mathbb{R}^2$.

$$A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

For each matrix:

1.1. Find all real eigenvalues.

1.2. Find a maximum set S of linearly independent eigenvectors