(Partial) Solution to Ex06, Q1

1. *Invariance under change of basis transformation*
In questions 1.1 and 1.2 please note that P is the change of basis matrix to a general basis, not necessarily the eigenbasis (if it even exists).

1.1. Show that the determinant is invariant under change of basis transformation. In other words, given that $B = P^{-1}AP$, show that $det(A) = det(B)$.
1.2. Show that the trace is invariant under change of basis transformation. In other words, show that $trace(A) = trace(B)$ for A, B as defined above.
Guidance: First prove that for any two matrices C, D it is true that $trace(CD) = trace(DC)$, and then use this fact to complete the proof.
1.3. Explain why from (1.1) and (1.2) you can conclude that for any diagonalizable matrix, the eigenvalues sum to the trace, and their product is the determinant.

Solution
1.1 We will solve this question in three method.
Method 1:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \underbrace{=}_{\substack{determinants\\ are\ scalars}} \det(P^{-1})\det(P)\det(A)$$

$$= \det\left(\underbrace{P^{-1}P}_{I}\right)\det(A) = \det(A)$$

Method 2:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \underbrace{=}_{\substack{determinants\\ are\ scalars}} \det(P^{-1})\det(P)\det(A)$$

$$= \frac{1}{\det(P)}\det(P)\det(A) = \det(A)$$

Method 3:

$$B = P^{-1}AP \quad \rightarrow \quad PB = AP \quad \rightarrow \det(PB) = \det(AP)$$

$$\det(P)\det(B) = \det(A)\det(P)$$

Since P is invertible $\det(P) \neq 0$, and we can divide both sides by $\det(P)$:

$$\det(B) = \det(A)$$

1.2 We first probe that $tr(AB) = tr(BA)$.
We use the formula of the general term $(AB)_{ij}$ for the case $i = j$:

$$tr(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$$

We can change the order of summation:

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ik}b_{ki}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki}a_{ik}\right) = \sum_{k=1}^{n}(BA)_{ii} = tr(BA)$$

Now, using this proposition we can see that: $tr(B) = tr(P^{-1}AP) = tr(APP^{-1}) = tr(A)$.