Solution to Ex06, Q6

1. A differential equation is an equation that relates a variable to its time derivative (i.e., the rate of change of this variable). An ordinary differential equation in one variable has the form: $\frac{dy}{dt} = ay$, and its solution is: $y = ce^{at}$, where c is some constant (that generally depends on the initial condition, $y(t = 0)$).

   You are given the following system of differential equations, which relates two time-dependent variables $y_1(t), y_2(t)$:

$$\frac{dy_1}{dt} = y_1 + 4y_2$$
$$\frac{dy_2}{dt} = 2y_1 + 3y_2$$

   1.1. Write the system in matrix form.

   1.2. What are the eigenvalues of the system?
   Use them to write down the solution vector to the system ($\bar{y}$) using the eigenvectors $\bar{v}_1$ and $\bar{v}_2$ (you don't have to explicitly find them at this point).

   1.3. Find an eigenvector associated with each of the eigenvalues.
   Write down the change of basis matrix P that represents a change of basis from the standard basis to the eigebasis.

   1.4. Use P and the solution vector $\bar{y}$ to find an explicit solution vector in the standard basis.

Solution

<u>Short remark on solving linear differential equations</u>

When solving the linear differential equation: $\frac{dy}{dt} = ay$, the solution is $y = ce^{at}$ (for some constant $c \in \mathbb{R}$). Because c can be any scalar, this solution is actually a family of solutions. If we want to know what c is, we need more information. If someone tells us what y is at time $t = 0$, we can find c. For example:

$$y(0) = 5$$
$$5 = ce^{0t} = c$$

So we found that $c = y(0)$. This is why we can write the family of solutions like this:

$$y(t) = y(0)e^{at}$$

The system in matrix form:

$$\frac{d\bar{y}}{dt} = \underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}}_{A} \bar{y}$$

The eigenvalues of the system:

$$det(A - \lambda I) = 0$$
$$(1 - \lambda)(3 - \lambda) - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda + 1)(\lambda - 5) = 0$$
$$\lambda_1 = -1 \quad \lambda_2 = 5$$

At this point we can solve the problem in two ways.

Method 1 – using the eigenvectors (represented in the standard space)

We know that the solution will be of the form:

$$\bar{y}(t) = e^{At}\bar{y}(0)$$

While we don't know what $\bar{y}(0)$ is, we know we can represent it as a linear combination of the eigenvectors, because the eigenvectors are a basis to the space:

$$y(0) = c_1\bar{v}_1 + c_2\bar{v}_2$$

So:

$$\bar{y}(t) = e^{At}(c_1\bar{v}_1 + c_2\bar{v}_2)$$

This is useful, because we know what e^{At} does to the eigenvector $\bar{v}_i$ – it just multiplies is by $e^{\lambda t}$:

$$\bar{y}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2$$
$$\bar{y}(t) = c_1 e^{-t}\bar{v}_1 + c_2 e^{5t}\bar{v}_2$$

Now let's go back and actually find the eigenectors. For $\lambda_1 = -1$:

$$\begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$2v_1 + 4v_2 = 0$$
$$v_1 = -2v_2$$

So we can set $v_2 = 5$ and get:

$$\bar{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For λ_2:

$$\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$v_1 - v_2 = 0$$

So we can set $v_2 = 1$ and get:

$$\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now, we can plug this back in the solution we found and get:

$$\bar{y}(t) = c_1 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is the solution up to some initial condition. If someone will tell us what was $\bar{y}(0)$ we could solve to find c_1 and c_2. For example, if $\bar{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\bar{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 e^{-0}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{0}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$-2c_1 + c_2 = 1$$
$$c_1 + c_2 = 0$$
$$c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$$

So:

$$\bar{y}(t) = -\frac{1}{3}e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \frac{1}{3}e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Method 2 – solving everything in the eigenbasis, and them changing back to the standard basis

In this method, we first solve in the eigenbasis. After finding the eigenvalues, be can say that in the eigembasis, every variable is solved separately (this was the whole idea behind changing to the eigenbasis). In this eigenbasis, the solution is:

$$\bar{y}_{eb} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{pmatrix}$$

Again, this is the solution up to some constants c_1, c_2 which we can find if someone tells us what $\bar{y}_{eb}(0)$ is.

Now, after finding the eigenvectors $\bar{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we can write down the change of basis matrix:

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$$

As we saw in the previous class, to change a vector in the eigenbasis, like $\bar{y}_{eb}$ to the standard basis, we multiply it by P from the left:

$$\bar{y} = P\bar{y}_{eb} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{pmatrix}$$

$$\bar{y} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{pmatrix} = \begin{pmatrix} -2c_1 e^{-t} + c_2 e^{5t} \\ c_1 e^{-t} + c_2 e^{5t} \end{pmatrix}$$

You can check and see that this is exactly what we found in the first method.