Linear Algebra for Neuroscience (76992)

Exercise 06 – Matrix diagonalization

1. *Invariance under change of basis transformation*
   In questions 1.1 and 1.2 please note that P is the change of basis matrix to a general basis, not necessarily the eigenbasis (if it even exists).

   1.1. Show that the determinant is invariant under change of basis transformation. In other words, given that $B = P^{-1}AP$, show that $det(A) = det(B)$.
   1.2. Show that the trace is invariant under change of basis transformation. In other words, show that $trace(A) = trace(B)$ for A, B as defined above.
   Guidance: First prove that for any two matrices C, D it is true that $trace(CD) = trace(DC)$, and then use this fact to complete the proof.
   1.3. Explain why from (1.1) and (1.2) you can conclude that for any diagonalizable matrix, the eigenvalues sum to the trace, and their product is the determinant.

2. Show that the identity matrix I_n (the identity matrix of size $n \times n$) is invariant under change of basis transformation.

3. In Ex05, Q2 you studied the matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

   Is A diagonalizable? If so, explain and find the change of basis matrix P such that $P^{-1}AP$ is diagonal.

4. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Calculate e^A.
   You may use what we learned in class about a general function of a matrix, $f(A)$.

5. True or false? Every diagonalizable matrix is also invertible. Explain.

6. A differential equation is an equation that relates a variable to its time derivative (i.e., the rate of change of this variable). An ordinary differential equation in one variable has the form: $\frac{dy}{dt} = ay$, and its solution is: $y = ce^{at}$, where c is some constant (that generally depends on the initial condition, $y(t = 0)$).

   You are given the following system of differential equations, which relates two time-dependent variables $y_1(t), y_2(t)$:

$$\frac{dy_1}{dt} = y_1 + 4y_2$$
$$\frac{dy_2}{dt} = 2y_1 + 3y_2$$

   6.1. Write the system in matrix form.

6.2. What are the eigenvalues of the system?
Use them to write down the solution vector to the system ($\bar{y}$) using the eigenvectors $\bar{v}_1$ and $\bar{v}_2$ (you don't have to explicitly find them at this point).

6.3. Find an eigenvector associated with each of the eigenvalues.
Write down the change of basis matrix P that represents a change of basis from the standard basis to the eigebasis.

6.4. Use P and the solution vector $\bar{y}$ to find an explicit solution vector in the standard basis.

7. Before the next class, please watch the video "Dot products and duality" in the "Essence of linear algebra" series (it should be chapter 9).
You can find it here (or search for the 3blue1brown channel on YouTube):
https://www.youtube.com/watch?v=LyGKycYT2v0

The next class could start with a short quiz about this video.

Additional questions

The following questions are not obligatory.

1. In the additional question from Ex05 you had the following real matrices defines some linear transformation on $\mathbb{R}^2$.

$$A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \ C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

Based on your answers there, determine if each matrix is diagonalizable. If it is, find a change of basis matrix P for which $D = P^{-1}AP$ is diagonal. Verify that D is indeed diagonal.