Linear Algebra for Neuroscience (76992)

Exercise 08 – Orthogonality

1. Consider the following three vectors:

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix}$$

1.1. Show that they are pairwise orthogonal.

1.2. Show that they form a basis of $\mathbb{R}^3$.

1.3. Let $\bar{u} = \begin{pmatrix} 4 \\ 14 \\ -9 \end{pmatrix}$. Write down $\bar{u}$ as a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$. Use the fact that the given basis is an orthogonal basis.

2. Cauchy-Shcwartz inequality

The Cauchy-Shcwartz inequality states that for any two vectors $\bar{v}, \bar{w}$:

$$\langle \bar{v}, \bar{w} \rangle^2 \le \langle \bar{v}, \bar{v} \rangle \langle \bar{w}, \bar{w} \rangle \qquad \text{or equivalently:} \qquad |\langle \bar{v}, \bar{w} \rangle| \le \left|\left|\bar{v}\right|\right| \left|\left|\bar{w}\right|\right|$$

Here one bar means "absolute value" and two bars mean "vector norm".

Verify the Cauchy-Schwartz inequality for $\bar{u} = (1,0,3,2)^T, \bar{v} = (4,1,0,1)^T$.

3. For this question, please watch the video "Lease squares – a concrete example" on moodle. In this video, we saw a concrete example for the "best" solution of an overdetermined system:

$$\hat{x} = (A^T A)^{-1} A^T \bar{b}$$

We found $\hat{x}_1$ and $\hat{x}_2$ by solving a simple system of two equations. Now, solve the system directly:

3.1. Find $(A^T A)^{-1}$.

3.2. Calculate $\hat{x}$.

4. Let $V = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$ be a set of nonzero pairwise-orthogonal vectors in $\mathbb{R}^n$.

4.1. Prove that V is linearly independent.

4.2. Assume that $k = n$. Do the vectors in V form a basis for $\mathbb{R}^n$?

5. Let A be a real symmetric matrix ($A^T = A$).

A is called **positive definite** if for any nonzero vector $\bar{x} \in \mathbb{R}^n$ it is true that:

$$\bar{x}^T A \bar{x} > 0$$

5.1. True or false? All eigenvalues of a positive definite matrix are positive.

5.2. What can you say about the angle between a vector before ($\bar{x}$) and after transformation by a positive definite matrix ($A\bar{x}$)? Note that $\bar{x}$ is a general vector, not necessarily an eigenvector.

Bonus question

In class we saw that for an underdetermined system $A\bar{x} = \bar{b}$, the projected vector $\hat{b}$ is:

$$\hat{b} = \underbrace{A(A^T A)^{-1} A^T}_{\substack{\text{the projection}\\ \text{matrix}}} \bar{b}$$

We can simplify this result and get:

$$\hat{b} = A(A^{-1}(A^T)^{-1})A^T \bar{b}$$
$$\hat{b} = \underbrace{AA^{-1}}_{I} \underbrace{(A^T)^{-1} A^T}_{I} \bar{b}$$
$$\hat{b} = \bar{b}$$

But this makes no sense, because this means that the projected vector is just the original vector. Something in this “proof” is flawed. Can you spot the flaw?