Linear Algebra for Neuroscience (76992)

Exercise 09 – Orthogonality and complex vectors and matrices

Orthogonality

1. The following vectors are a basis for a subspace $V \subset \mathbb{R}^4$:

$$\bar{v}_1 = (1,1,1,1)^T \quad \bar{v}_2 = (1,2,4,5)^T \quad \bar{v}_3 = (1,-3,-4,-2)^T$$

Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

2. Orthogonal matrices

2.1. Prove that the determinant of an orthogonal matrix is either +1 or -1.

2.2. Prove that the product of two orthogonal matrices is an orthogonal matrix.

Complex vectors and matrices

3. Let $A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

3.1. Is A diagonalizable over the real numbers $\mathbb{R}$? Justify your answer.

3.2. Is A diagonalizable over the complex numbers $\mathbb{C}$? Justify your answer.

4. Show that $A = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i & \frac{2}{3}i \\ -\frac{2}{3}i & -\frac{1}{3} - \frac{2}{3}i \end{pmatrix}$ is unitary.

5. True or false?

5.1. All the diagonal entries of an Hermitian matrix are real.

5.2. Every Hermitian matrix is also a normal.

Definition: A complex matrix is said to be normal if it commutes with A^H:

$$AA^H = A^H A$$

A matrix is normal if and only if it is diagonalizable by a unitary matrix U:

$$UAU^{-1} = \Lambda$$

6. Before the next class, please watch the video “Abstract vector spaces” in the “Essence of linear algebra” series.

You can find it here (or search for the 3blue1brown channel on YouTube):

https://www.youtube.com/watch?v=TgKwz5Ikpc8&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=14

The next class could start with a short quiz about this video.

Additional questions

The following questions are not obligatory.

1. Let $\bar{u} = \begin{pmatrix} 7 - 2i \\ 2 + 5i \end{pmatrix}, \bar{v} = \begin{pmatrix} 1 + i \\ -3 - 6i \end{pmatrix}$

1.1. Find $\bar{u} + 2i\bar{v}$

1.2. Find $\bar{u}^{*T}\bar{v}$

1.3. Find $|\bar{u}|$

2. Let $A = \begin{pmatrix} 3 - 5i & 2 + 4i \\ 6 + 7i & 1 + 8i \end{pmatrix}$. Find A^H.

3. Show that the following matrix is normal:

$$A = \begin{pmatrix} 2 + 3i & 1 \\ i & 1 + 2i \end{pmatrix}$$